PLEASE HAND IN

UNIVERSITY OF TORONTO
Faculty of Arts and Science

Term test #1

CSC 236H1
Duration — 50 minutes

PLEASE HAND IN

Last Name: ______________________________

First Name: ______________________________

Do **not** *turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This test consists of 3 questions on 4 pages (including this one). *When you receive the signal to start, please make sure that your copy of the test is complete.*

Please answer questions in the space provided. You will earn 20% for any question you leave blank or write "I cannot answer this question," on.

Good Luck!

Question 1. [10 MARKS]

Consider the function:

$$a(n) = \begin{cases} 1 & \text{if } n = 1 \\ \left[a(\lfloor\sqrt{n}\rfloor)\right]^2 + 2a(\lfloor\sqrt{n}\rfloor) & \text{if } n > 1 \end{cases}$$

Use any reasonable variant of Complete Induction to prove that $a(n)$ is divisible by 5 for all natural numbers $n \geq 4$. For clarity, be sure to:

1. Label your Inductive Hypothesis (IH)
2. When you use your IH, mention which numbers you are using it for, and why the IH may be used, e.g. "...since both $n-1$ and $n-2$ are between..."

Sample solution: Define $P(n)$: "There is some natural number k such that $a(n) = 5k$." I prove that for all natural numbers n greater than 3, $P(n)$ holds.

Inductive step: Assume n is a generic natural number greater than 3, and that for all $4 \leq k < n$, $P(k)$ holds (Inductive Hypothesis, IH). I need to show $P(n)$ follows.

For convenience, I verify that $a(2) = a(3) = 3$.

Case 0, $n \in \{4, 5, \ldots, 8\}$: Then

$$\begin{aligned} a_n &= a(2)^2 + 2a(2) && \# \text{ since } \lfloor\sqrt{n}\rfloor = 2 \\ &= 3^2 + 6 = 15 && \# \text{ since } a_2 = 3 \end{aligned}$$

So $P(n)$ holds in this case.

Case 1, $n \in \{9, 10, \ldots, 15\}$: Then

$$\begin{aligned} a_n &= a(3)^2 + 2a(3) && \# \text{ since } \lfloor\sqrt{n}\rfloor = 3 \\ &= 3^2 + 6 = 15 && \# \text{ since } a_3 = 3 \end{aligned}$$

So $P(n)$ holds in this case.

Case 2, $n \geq 16$: Then $4 \leq \lfloor\sqrt{n}\rfloor < n$, so

$$\begin{aligned} a(n) &= a(\lfloor\sqrt{n}\rfloor)^2 + 2a(\lfloor\sqrt{n}\rfloor) && \# \text{ since } n > 1 \\ &= (5k)^2 + 2(5k) \text{ for some } k \in \mathbb{N} && \# \text{ IH, since } 4 \leq \lfloor\sqrt{n}\rfloor < n \\ &= 5(5k^2 + 2k) = 5k_1, k_1 \in \mathbb{N} && \# \text{ since } \mathbb{N} \text{ closed under } +, \times \end{aligned}$$

So $P(n)$ holds in this case.

In each possible case, $P(n)$ holds, so $[\forall k \in \{4, \ldots, n-1\}, P(k)] \rightarrow P(n)$.

I conclude, by Complete Induction, that $\forall n \in \mathbb{N} - \{0, 1, 2, 3\}, P(n)$.

Question 2. [10 MARKS]

Use any reasonable variant of Simple Induction on i to prove that if m is a natural number greater than 1, then for each natural number i: $(m-1)^{2i} = mk+1$, for some natural number k.

Sample solution: Define $P(i) : (m-1)^{2i} = mk+1$, for some natural number k. I prove that if m is a natural number greater than 1, then $\forall i \in \mathbb{N}, P(i)$, using Simple Induction.

Assume m is a typical natural number greater than 1.

Base case: If $i = 0$, then $(m-1)^{2i} = 1 = 0 \times m + 1$, and $0 \in \mathbb{N}$, so $P(0)$ holds.

Inductive step: Assume i is a typical natural number, and that $P(i)$ holds, that is $(m-1)^{2i} = mk$ for some natural number k (Inductive Hypothesis, IH). I must show that $P(i+1)$ follows, that is $(m-1)^{2(i+1)} = mk_1$ for some $k_1 \in \mathbb{N}$.

I have:

$$
\begin{aligned}
(m-1)^{2(i+1)} &= (m-1)^{2i} \times (m-1)^2 \qquad \text{\# factoring...} \\
&= (km+1)(m^2-2m+1), \text{ some } k \in \mathbb{N} \\
&\qquad \text{\# by IH we have } P(i) \text{, and expanding } (m-1)^2 \\
&= (km+1)(m(m-2)+1 = m\left((m-2)(km+1)+k\right)+1 \\
&= k_1 m + 1, \text{ where } k_1 \in \mathbb{N} \\
&\qquad \text{\#}\mathbb{N} \text{ closed under } +, \times \text{ and } m-2 \geq 0 \text{ since } m \geq 2 \\
&\qquad \text{\# so } (m-2)(km+1)+k \in \mathbb{N}
\end{aligned}
$$

So $P(i+1)$ follows from $P(i)$.

I conclude, by Simple Induction, $\forall \in \mathbb{N}, P(i)$

Since m is a generic natural number greater than 1, I have shown that for all natural numbers greater than 1, $\forall \in \mathbb{N}, P(i)$.

Question 3. [10 MARKS]

Define the set of arithmetic expressions, $\mathcal{E}$, as the smallest set such that:

1. Variables x, y, and z are elements of $\mathcal{E}$
2. If e_1 and e_2 are elements of $\mathcal{E}$, then so are $[e_1 + e_2], [e_1 - e_2], [e_1 \times e_2]$, and $[e_1 \div e_2]$.

For example, under this definition $x, y, [x + x]$, and $[x + [y - x]]$ are elements of $\mathcal{E}$.

Define $V(e)$, the number of variables in e, $O(e)$, the number of operators in e, and $B(e)$, the number of brackets in e by:

1. If $e \in \{x, y, z\}$, then $V(e) = 1$, $O(e) = B(e) = 0$.
2. If $e_1, e_2 \in \mathcal{E}$, and $e = [e_1 \odot e_2]$, where $\odot \in \{+, \times, -, \div\}$, then

$$\begin{aligned} V(e) &= V(e_1) + V(e_2) \\ O(e) &= O(e_1) + O(e_2) + 1 \\ B(e) &= B(e_1) + B(e_2) + 2 \end{aligned}$$

Use Structural Induction to prove that, for all $e \in \mathcal{E}$ $B(e) = V(e) + O(e) - 1$. Be sure to note where you use your induction hypothesis, and why you are justified in using it.

Sample solution: Define $P(e) : B(e) = V(e) + O(e) - 1$. I will prove $\forall e \in \mathcal{E}, P(e)$, using Structural Induction.

Base case: If $e \in \{x, y, z\}$, then $B(e) = 0$, $V(e) = 1$, and $O(e) = 0$, from the definition, so $B(e) = 0 = 1 + 0 - 1 = V(e) + O(e) - 1$, as claimed. So $P(e)$ holds for the basis.

Inductive step: Assume e_1 and e_2 are typical elements of $\mathcal{E}$ and that $P(e_1)$ and $P(e_2)$ are true (Inductive Hypothesis, IH). I must show that this implies $P([e_1 \odot e_2])$, for $\odot \in \{+, \times, -, \div\}$.

I have:

$$\begin{aligned} B([e_1 \odot e_2]) &= B(e_1) + B(e_2) + 2 && \text{\# from definition of } V(e) \\ &= V(e_1) + O(e_1) - 1 + V(e_2) + O(e_2) - 1 + 2 && \text{\# by IH, since I assumed } P(e_1) \text{ and } P(e_2) \\ &= [V(e_1) + V(e_2)] + [O(e_1) + O(e_2) + 1] - 1 && \text{\# regroup...} \\ &= V([e_1 \odot e_2]) + O([e_1 \odot e_2]) - 1 && \text{\# by definitions of } V(e) \text{ and } O(e) \end{aligned}$$

So $B([e_1 \odot e_2]) = V([e_1 \odot e_2]) + O([e_1 \odot e_2]) - 1$, that is $P([e_1 \odot e_2])$.

I conclude, by Structural Induction, $\forall e \in \mathcal{E}, P(e)$.

1: ______/10

2: ______/10

3: ______/10

TOTAL: ______/30