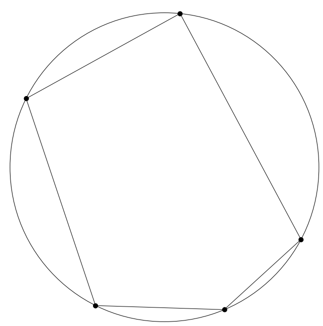
Assignment 1
CSC236, Fall 2008

The following exercises are meant to give you experience with several flavours of induction. These proof techniques are often used in reasoning about computation, and it is useful to be familiar with them.

1. Draw five distinct points on a circle. Draw line segments between adjacent pairs of points, yielding a pentagon. The sum of the interior angles of this figure is 3π radians (540 degrees). Use induction to prove that for $n > 2$ distinct points on a circle, if line segments are drawn between adjacent pairs of points, then the sum of the interior angles is $(n-2)\pi$ radians. You may take it as a given fact that the sum of interior angles of a triangle is π radians.



2. A restaurant has two meals, linguini and stew. They decide to prepare a repeating cycle of all four possible lunch menus: a menu that contains neither meal (on their day off), a menu that contains both linguini and stew, a linguini-only menu, and a stew-only menu. The schedule has the feature that the menu for any two consecutive days differs by exactly one meal (this reduces stress in the kitchen):

$$\ldots, \{L\}, \{L, S\}, \{S\}, \{\}, \{L\}, \{L, S\}, \{S\}, \{\}, \ldots$$

   Design a repeating cycle of all eight possible lunch menus for three meals — linguini, stew, and panini — that uses all eight possible menus comprised of them, and ensures that any two consecutive days differ by exactly one meal.

   Propose a technique for designing repeating cycles of all possible 2^n menus for n meals that ensures that on any two consecutive days the menus differ by exactly one meal. Use simple induction to prove that your technique works.

3. The golden ratio, ϕ, has the following property:

$$\phi = \frac{1}{\phi - 1}$$

(a) Assume (for the sake of contradiction) that n_1 and n_2 are two natural numbers with the property that $n_1/n_2 = \phi$. What does the assumption imply about $n_2/(n_1 - n_2)$? Continue this line of argument, and use the Principle of Well-Ordering, to derive a contradiction.

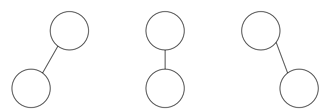
(b) Use the previous part to prove that $\sqrt{5}$ is irrational.

4. Define the set of ternary trees, $\mathcal{T}_\ni$, as:

   (a) The empty tree, Λ, is an element of $\mathcal{T}_3$.

   (b) If trees T_L, T_M, T_R are elements of $\mathcal{T}_\ni$ having no nodes in common, and R is a node that is not in T_L, T_M or T_R, then $T = (R, T_L, T_M, T_R)$ is an element of $\mathcal{T}_3$. We call T_L the left subtree, T_M the middle subtree, and T_R the right subtree, of T.

   Consider two trees equivalent if (a) they are both Λ (the empty tree), or (b) they have equivalent left subtrees, equivalent middle subtrees, and equivalent right subtrees. So there is one ternary tree with zero nodes: Λ, and one ternary tree with one node: $(R, \Lambda, \Lambda, \Lambda)$. For arbitrary natural number n, find a (possibly recursive) formula for the number of non-equivalent ternary trees with n nodes. Prove your formula is correct.



Artist's impression of the three non-equivalent ternary trees with two nodes.