CSC236 Tutorial Exercises, Mar 2/3
(Sample Solution)

1. A non-empty array $A$ with integer entries has the property that no odd number occurs at a lower index than an even number. Consider the following algorithm for finding the highest index of an even element between indices $b$ and $e$, inclusive, or $b-1$ if $A$ has no elements that are even numbers.

   recHighestEven($A$, $b$, $e$):
   
   if $b == e$:
     if $A[b] \% 2 == 0$ : return $b$
     else: return $b - 1$
   else:
     $m = (b + e) // 2$ # midpoint
     if $A[m+1] \% 2 == 1$:
       return recHighestEven($A$, $b$, $m$)
     else:
       return recHighestEven($A$, $m+1$, $e$)

(a) State preconditions and postconditions for this algorithm.

Preconditions:
- $A$ is non-empty with integer elements
- $A$ has the property that no odd number occurs at a lower index than an even number
- $b, e \in \mathbb{N}$
- $0 \leq b \leq e < \text{length}(A)$

Postconditions: recHighestEven terminates and returns $k$ such that
- $b - 1 \leq k \leq e$
- $\forall i, b \leq i \leq k$, $A[i]$ is even
- $\forall j, k < j \leq e$, $A[j]$ is odd

(b) Prove the algorithm is correct by showing that (preconditions) $\rightarrow$ (termination $\land$ postconditions).

Proof by induction:
Let $n = e - b + 1$.

Inductive Step: Let $n > 1$.

Assume $H(n): \forall i, 1 \leq i < n$, suppose the algorithm terminates and the postconditions hold after execution for all inputs of size $i$ that satisfy the preconditions.

Show $H(n) \rightarrow C(n)$: the algorithm terminates and the postconditions hold after execution for all inputs of size $n$ that satisfy the preconditions.
Consider a call to \texttt{recHighestEven}(A, b, e) with \( n \geq 2 \). The test \( b == e \) fails since \( b < e \), and so the next line executed is computation of \( m \).

**Case:** \( A[m+1] \) is odd

Here, the first recursive call \texttt{recHighestEven}(A, b, m) is made. Since \( b < e \) then \( m < e \), and \( m - b + 1 < e - b + 1 \). Thus, by \( H(n) \), this recursive call terminates and returns \( k \) that satisfies by the postconditions. That is,

(i) \( b - 1 \leq k \leq m \)
(ii) \( \forall i, b \leq i \leq k, A[i] \) is even
(iii) \( \forall j, k < j \leq m, A[j] \) is odd

Since the recursive call terminates and no other statements are executed, then the algorithm terminates in this case. It remains to be shown that the postconditions are satisfied.

- \( b - 1 \leq k \leq e \), by (i) and \( m < e \)
- \( \forall i, b \leq i \leq k, A[i] \) is even, by (ii)
- \( \forall j, k < j \leq e, A[j] \) is odd, by (iii), this case \( (A[m+1] \) odd \), and the property of \( A \)

**Case:** \( A[m+1] \) is even

Here, the second recursive call \texttt{recHighestEven}(A, m+1, e) is made. Since \( b < e \) then \( m + 1 > b \), and \( e - (m + 1) + 1 < e - b + 1 \). Thus, by \( H(n) \), this recursive call terminates and returns \( k \) that satisfies by the postconditions. That is,

(i) \( m + 1 - 1 \leq k \leq e \)
(ii) \( \forall i, m + 1 \leq i \leq k, A[i] \) is even
(iii) \( \forall j, k < j \leq e, A[j] \) is odd

Since the recursive call terminates and no other statements are executed, then the algorithm terminates in this case. It remains to be shown that the postconditions are satisfied.

- \( b - 1 \leq k \leq e \), by (i) and \( m + 1 > b \)
- \( \forall i, b \leq i \leq k, A[i] \) is even, by (ii), this case \( (A[m+1] \) is even \), and the property of \( A \)
- \( \forall j, k < j \leq e, A[j] \) is odd, by (iii)

So, \( H(n) \to C(n) \) for \( n > 1 \).

**Base case:** Let \( n = e - b + 1 = 1 \), and assume the preconditions are satisfied.

Then, the algorithm terminates, since there are no loops or recursive calls, and returns \( k \) such that

- \( b - 1 \leq k \leq e \), because either \( k = b - 1 \) or \( k = b = e \)
- \( \forall i, b \leq i \leq k, A[i] \) is even, because if \( b \) is returned, then \( A[b] \) is even
- \( \forall j, k < j \leq e, A[j] \) is odd, because if \( b - 1 \) is returned, then \( A[b] \) is odd

**Conclusion:** Thus, in all cases, \texttt{recHighestEven} is correct.