1. (7 pts) Use simple induction to prove that $3^n > n^3 + n$ for all natural numbers $n > 3$. You may use the fact that $(n + 1)^3 = n^3 + 3n^2 + 3n + 1$. For full marks your proof must clearly indicate any necessary base case(s), the inductive hypothesis, and where the inductive hypothesis is used.

**sample solution:** Define $P(n) : 3^n > n^3 + n$. Proof by simple induction that $\forall n \in \mathbb{N} \setminus \{0, 1, 2, 3\}, P(n)$.

base case: $3^4 = 81 > 68 = 4^3 + 4$, so $P(4)$ holds.

inductive step: Let $n$ be an arbitrary natural number greater than 3. Assume $H(n)$, that is $3^n > n^3 + n$.

derive inductive conclusion $C(n)$: that is, $3^{n+1} > (n + 1)^3 + (n + 1)$.

$$3^{n+1} = 3 \times 3^n > 3(n^3 + n) \quad \# \text{by } H(n)$$
$$= 3n^3 + 3n = n^3 + n^2 + n^2 n + 3n \quad \# \text{expanding}$$
$$> n^3 + 3n^2 + 4n + 2 \quad \# n > 3, n^2 > 9 > 4, n > 3 > 2/3$$
$$= (n + 1)^3 + (n + 1) \quad \# \text{re-writing}$$

So $C(n)$ \blacksquare

2. (12 pts) Read the definition of function $f$:

$$f(n) = \begin{cases} 
1 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
2 & \text{if } n = 2 \\
(f(\lfloor n/3 \rfloor))^3 + f(\lfloor n/3 \rfloor) & \text{if } n > 2
\end{cases}$$

Use complete induction to prove that for every natural number $n$ larger than 1, $f(n)$ is even. For full marks your proof must clearly indicate any necessary base case(s), the inductive hypothesis, where the inductive hypothesis is used, and why it is valid to use it there.

**sample solution:** Define $P(n)$: $f(n)$ is a multiple of 2. Proof by complete induction that $\forall n \in \mathbb{N} \setminus \{0, 1\}, P(n)$.

inductive step: Let $n \in \mathbb{N} \setminus \{0, 1\}$. Assume $H(n)$: $\forall i \in \mathbb{N}$ if $2 \leq i < n$ then $f(i)$ is a multiple of 2.

must show inductive hypothesis $C(n)$: $f(n)$ is a multiple of 2.

Base case, $n = 2$: Then $f(n) = 2$, from definition, and 2 is notoriously a multiple of 2.
Base case, \( n \in \{3, 4, 5\} \): Then
\[
f(n) = (f(\lfloor n/3 \rfloor))^2 + f(\lfloor n/3 \rfloor) \quad \# \text{from definition of } f(n), \ n > 2
\]
\[
= (f(1))^2 + f(1) \quad \# \ |n/3| = 1 \text{ for } n \in \{3, 4, 5\}
\]
\[
= 1 + 1 = 2 \quad \# \text{from definition of } f(1), \ 2 \text{ is a multiple of } 2
\]

Case \( n > 5 \): \( f(\lfloor n/3 \rfloor) \) is a multiple of 2, by \( H(n) \), since \( 2 \leq \lfloor n/3 \rfloor < n \) when \( n \) is at least 6. Let \( k \in \mathbb{N} \) such that \( f(\lfloor n/3 \rfloor) = 2k \)
\[
f(n) = (f(\lfloor n/3 \rfloor))^2 + f(\lfloor n/3 \rfloor) \quad \# \text{from definition of } f(n), \ n > 2
\]
\[
= (2k)^2 + 2k = 2(2k^2 + k)
\]
\[
\# \text{a multiple of 2 since } 2, k \in \mathbb{N} \text{ and } \mathbb{N} \text{ is closed under } +, \times
\]

In every possible case \( C(n) \) follows

3. (10 pts) Define the set of non-empty binary trees \( B^* \) (not full binary trees) as the smallest set such that:

a) A solitary node with no edges is an element of \( B^* \).

b) If \( t_1, t_2 \in B^* \) and \( n \) is a single node, then the following three trees are also elements of \( B^* \):
   i. the tree formed with root \( n \) and an edge to left subtree \( t_1 \);
   ii. the tree formed with root \( n \) and an edge to right subtree \( t_2 \);
   iii. the tree formed with root \( n \), an edge to left subtree \( t_1 \), and an edge to right subtree \( t_2 \).

For \( t \in B^* \) define \( N(t) \) as the number of nodes in \( t \), and \( E(t) \) as the number of edges in \( t \). Use structural induction to prove that for all \( t \in B^* \), \( N(t) = E(t) + 1 \). For full marks your proof must clearly indicate any base case(s), the inductive hypothesis, and where the inductive hypothesis is used.

sample solution: Define \( P(t) \): \( N(t) = E(t) + 1 \). Proof by structural induction that \( \forall t \in B^* \), \( P(t) \)

verify basis: A solitary node \( t \) with no edges has \( N(t) = 1 = 0 + 1 = E(t) + 1 \). So the claim holds for the basis.

inductive step: Let \( t_1, t_2 \) be arbitrary elements of \( B^* \) and \( n \) be a single node. Assume \( H(\{t_1, t_2\}) \), that is \( P(t_1) \) and \( P(t_2) \).

must show \( C(\{t_1, t_2\}) \): that is, any tree \( t \) in \( B^* \) formed from root \( n \), with subtrees \( t_1, t_2 \) satisfies \( P(t) \). There are 3 cases to consider:
Case i: If $t$ is a tree formed from root $n$ with left subtree $t_1$, then $N(t) = N(t_1) + 1$, since $n$ provides one new node. Also $E(t) = E(t_1) + 1$, since there is a new edge from $n$ to its subtree. Summing up:

\[
N(t) = N(t_1) + 1 = (E(t_1) + 1) + 1 \quad \# \text{ by } H\{t_1, t_2\} \\
= E(t) + 1 \quad \# \text{ since } n \text{ add one node.}
\]

$P(t)$ follows in this case.

Case ii: If $t$ is a tree formed from root $n$ with right subtree $t_2$, the argument is the same as Case ii with $t_1$ replace by $t_2$, left replaced by right, and $P(t)$ follows in this case.

Case iii: If $t$ is a tree formed from room $n$ with left subtree $t_1$ and right subtree $t_2$, then $N(t)$ is the sum of the nodes in the two subtrees, plus one for $n$ itself, or $N(t) = N(t_1) + N(t_2) + 1$. $E(t)$ is the sum of the edges in the two subtrees, plus two edges connecting $t$ to each subtree, or $E(t) = E(t_1) + E(t_2) + 2$. Summing up:

\[
N(t) = N(t_1) + N(t_2) + 1 = \\
= (E(t_1) + 1) + (E(t_2) + 1) + 1 \quad \# \text{ by } H\{t_1, t_2\} \\
= (E(t_1) + E(t_2) + 2) + 1 = E(t) + 1
\]

$P(t)$ follows in this case.

In every possible case $P(t)$ follows, that is $C\{t_1, t_2\}$.