Outline

Structural Induction

Well-ordering
Define sets recursively...
...so as to use induction on them later

Consider the following set $S$:

1. $0 \in S$
2. $n \in S \implies n + 1 \in S$.
3. $S$ contains nothing else (i.e. $S$ is the smallest such set)
Define sets recursively...

...so as to use induction on them later

This is one way define the natural numbers $\mathbb{N}$. That is, $S \equiv \mathbb{N}$.

Every element of $S$ is a natural number (by construction, and smallest).

Every natural number is an element if $S$ (every natural number is 1 more than another, except 0).

By smallest, we mean $\mathbb{N}$ has no proper subsets that satisfy these conditions. If we leave out smallest, what other sets satisfy the definition?
What can you do with it?

We defined the simplest natural number (0) and the rule to produce new natural numbers from old (add 1). Proofs using Simple Induction work by showing that 0 has some property, and then that the rule to produce natural numbers preserves the property, that is

1. Show that $P(0)$ is true for basis, 0
2. Prove that $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n + 1)$.

The same structure applies if our set is defined differently, e.g.

$S = \{ n \in \mathbb{N} \mid n \geq 2 \}$
Other Structurally-Defined Sets

A set of expressions $\mathcal{E}$

Define $\mathcal{E}$: The smallest set such that

- $x, y, z \in \mathcal{E}$
- $e_1, e_2 \in \mathcal{E} \Rightarrow (e_1 + e_2), (e_1 - e_2), (e_1 \times e_2)$, and $(e_1 \div e_2) \in \mathcal{E}$.

Form some expressions in $\mathcal{E}$. Count the number of variables (symbols from $\{x, y, z\}$) and the number of operators (symbols from $\{+, \times, \div, -\}$). Make a conjecture about the number of variables and the number of operators in an expression in $\mathcal{E}$.
Structural Induction outline

\( P(e) : \text{vars}(e) = \text{op}(e) + 1 \)

To prove that a property is true for all \( e \in \mathcal{E} \), parallel the recursive set definition:

**Verify Base Case(s):** Show that the property is true for the simplest members, \( \{x, y, z\} \), that is show \( P(x) \), \( P(y) \), and \( P(z) \).

**Inductive Step:** Let \( e_1 \) and \( e_2 \) be arbitrary elements of \( \mathcal{E} \). Assume \( H(\{e_1, e_2\}) : P(e_1) \) and \( P(e_2) \), that is \( e_1 \) and \( e_2 \) have the property.

**Show that \( C(\{e_1, e_2\}) \) follows:**

All possible combinations of \( e_1 \) and \( e_2 \) have the property, that is

\( P((e_1 + e_2)) \), \( P((e_1 - e_2)) \),
\( P((e_1 \times e_2)) \), and \( P((e_1 \div e_2)) \).
Structural induction

$P(e) : \text{vars}(e) = \text{op}(e) + 1$

Prove $\forall e \in \mathcal{E}, P(e)$ by Structural induction
Structural induction

\[ P(e) : \text{vars}(e) = \text{op}(e) + 1 \]
More structural induction

Define the height of \(x\), \(y\), or \(z\) as 0, and \(h((e_1 \odot e_2))\) as \(1 + \max(h(e_1), h(e_2))\), if \(e_1, e_2 \in \mathcal{E}\) and \(\odot \in \{+, \times, \div, -\}\). What’s the connection between the number of variables and the height?
More structural induction

\[ P(e) : \text{vars}(e) \leq 2^{h(e)} \]
More structural induction

$P(e) : \text{vars}(e) \leq 2^{h(e)}$
Principle of Well-Ordering

Every non-empty subset of $\mathbb{N}$ contains a smallest element.

1. Applies to both finite and infinite subsets of $\mathbb{N}$
2. A property of $\mathbb{N}$ – does not apply to sets like $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
Using Well-Ordering

Prove that $\forall n \in \mathbb{N}, \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$
Using Well-Ordering

Prove that $\forall n \in \mathbb{N}, \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$
Using Well-Ordering

Prove \( \forall m, n \in \mathbb{N}, n > 0, \exists r, \exists q, \text{ s.t. } r < n \text{ and } m = qn + r \)
Using Well-Ordering

Prove $\forall m, n \in \mathbb{N}, n > 0, \exists r, \exists q, \text{s.t. } r < n$ and $m = qn + r$
Well-Ordering proof outline

Template for a proof of $\forall n, P(n)$ using Well-Ordering:

1. For a contradiction, suppose $\exists n \in \mathbb{N}, \neg P(n)$.
2. Then, set $S = \{n \in \mathbb{N} : \neg P(n)\}$ is not empty.
3. By Well-Ordering, $S$ contains a smallest element $k$.
4. At this point, we know:
   - (i) $\neg P(k)$ (because $k \in S$)
   - (ii) $P(i), \forall i \in \{0, \ldots, k-1\}$ (because $k$ is smallest in $S$)
5. Use both facts to derive a contradiction.
6. Hence, by contradiction, $\forall n \in \mathbb{N}, P(n)$. 
How is Well-Ordering related to Induction?

- It turns out that Simple Induction, Complete Induction, and Well Ordering are all equivalent to each other.
- You can use any one of them to conclude the other two.
- See Theorem 1.1 on p. 19 of the course notes
Notes