CSC236 Week 2: Outline

Principle of Complete Induction

Examples of Complete Induction
Prove by Induction...

Every natural number greater than 1 has a prime factorization

\[ 2 \Rightarrow 2 \]
\[ 3 \Rightarrow 3 \]
\[ 4 \Rightarrow 2 \cdot 2 \]
\[ 5 \Rightarrow 5 \]
\[ 6 \Rightarrow 2 \cdot 3 \]
\[ 7 \Rightarrow 7 \]
\[ 8 \Rightarrow 2 \cdot 2 \cdot 2 \]

Given the prime factor (P.F.) of 8, what is the PF of 9?

Doesn't tell us anything about 9

\[ 9 \Rightarrow 3 \cdot 3 \]
Prove by Induction...

Every natural number greater than 1 has a prime factorization

How does the factorization of 8 help with the factorization of 9?

We need another flavour of induction
The Principle of Complete Induction

Simple Induction is a special case of Comp. I.

\[(\forall n \in \mathbb{N}, \langle P(0), \ldots, P(n-1) \rangle \Rightarrow P(n)) \Rightarrow \forall n \in \mathbb{N}, P(n)\]

If all the previous cases always implies the current case then all cases are true.
Complete Induction Outline

**Inductive Step:** State Inductive Hypothesis $H(n)$

**Derive Conclusion $C(n):** Show that $C(n)$ follows from $H(n)$, indicating where you use $H(n)$ and why that is valid

**Verify Base Case(s):** Verify that the claim is true for any cases not covered in the inductive step

Wait! Isn’t that the same outline as simple induction?

Yes, we just modify the inductive hypothesis, $H(n)$ so that it assumes the main claim for every natural number from the starting point up to $n - 1$, and the conclusion, $C(n)$, is now the main claim for $n$. 

will cover a larger range
Is \( f(n) \) a multiple of 3?

\[
 f(n) = \begin{cases} 
 1 & \text{if } n \leq 1 \\
 [f(\lfloor \sqrt{n} \rfloor)]^2 + 2f(\lfloor \sqrt{n} \rfloor) & \text{if } n > 1 
\end{cases}
\]

Check a few cases, and make a conjecture:

\[
\begin{align*}
 f(0) &= 1 \\
 f(1) &= 1 \\
 f(2) &= f(1)^2 + 2 \cdot f(1) = 3 \\
 f(3) &= 3 \quad (\text{because } \lfloor \sqrt{3} \rfloor = \lfloor \sqrt{2} \rfloor) \\
 f(4) &= f(2)^2 + 2 \cdot f(2) = 3^2 + 6 = 15 \\
 f(5) &= 15 \\
 f(6), f(7), f(8) &= 15 \\
 f(9) &= f(3)^2 + 2 \cdot f(3) = 15 \\
 f(10) &= f(4)^2 + 2 \cdot f(4) \\
 &= 225 + 30 = 255 \\
\end{align*}
\]

Conjecture: \( f(n) \) is a multiple of 3 for all \( n \in \mathbb{N} \) s.t. \( n \geq 2 \).
For all natural numbers \( n > 1 \), \( f(n) \) is a multiple of 3?

use the complete induction outline

Inductive Step: Let \( n \in \mathbb{N}, n \geq 2 \)

Assume \( H(n) \): for \( i \in \mathbb{N}, 2 \leq i < n \)

\( f(i) \) is a multiple of 3

Show \( H(n) \Rightarrow C(n) : f(n) \) is a multiple of 3.

(Computing \( f(2), f(3) \) depends on \( f(1) \), which is not covered in \( H(n) \))

Let \( n \geq 4 \). \( f(n) = \left( f(\lfloor \sqrt{n} \rfloor) \right)^2 + 2 \cdot f(\lfloor \sqrt{n} \rfloor) \)

\( 2 \leq \sqrt{n} \) (because \( n \geq 4 \))

\( \sqrt{n} < n \) (because \( n \geq 4 > 1 \))

\( 2 \leq \lfloor \sqrt{n} \rfloor \leq \sqrt{n} < n \)

Thus, \( f(\lfloor \sqrt{n} \rfloor) \) is a multiple of 3 by \( H(n) \).
For all natural numbers $n > 1$, $f(n)$ is a multiple of 3?

use the complete induction outline

Let $k \in \mathbb{N}$ s.t. $f(L\sqrt{n}) = 3 \cdot k$ (since $f(L\sqrt{n})$ is a multiple of 3)

So $f(n) = [f(L\sqrt{n})]^2 + 2 \cdot f(L\sqrt{n})$

$= (3 \cdot k)^2 + 2 \cdot 3 \cdot k = 9k^2 + 6k = 3(3k^2 + 2k)$

$2, 3, k \in \mathbb{N} \Rightarrow (3k^2 + 2k) \in \mathbb{N}$

so $f(n)$ is a multiple of 3, for $n \in \mathbb{N}, n \geq 4$.

Verify base cases:

$f(2) = [f(1)]^2 + 2 \cdot f(1) = 3 = 3 \times 1$

$f(3) = [f(1)]^2 + 2 \cdot f(1) = 3 = 3 \times 1$

so the claim holds for 2 and 3.

Conclude: $f(n)$ is a multiple of 3 $\forall n \in \mathbb{N}, n \geq 2$
Zero Pair Free Binary Strings (zpfb(n))

Denote by $zpfb(n)$ the number of binary strings of length $n$ that contain no pairs of adjacent zeros. What is $zpfb(n)$ for the first few natural numbers $n$?

$zpfb(0) = 1$ (""" - the empty string)

$zpfb(1) = 2$ ("0", "1")

$zpfb(2) = 3$ ("01", "11", "10")

$zpfb(3) = 5$ ("011", "111", "101", "010", "110")

$zpfb(4) = 8$ ("0111", "1111", "1011", "0101", "1101", "1010"

Claim: $zpfb(n)$ is $2p(n)$. (The number of ZPFBS of length $n$ is represented by $zp(n)$.)

\[
zp(n) = \begin{cases} 
1 & n=0 \\
2 & n=1 \\
zp(n-1) + zp(n-2) & n \geq 2 
\end{cases}
\]
What is \( zpfbs(n) \)?

use the complete induction outline

**Inductive Step:** Let \( n \in \mathbb{N} \).

Assume \( H(n) \): \( \forall \ i \in \mathbb{N}, \ 0 \leq i < n \), \( zp(i) \) is the number of ZPFBS of length \( i \).

Show \( H(n) \rightarrow C(n) \): \( zp(n) \) is the number of ZPFBS of length \( n \).

Let \( n \geq 2 \). Then \( 0 \leq n-1 < n \) and \( 0 \leq n-2 < n \).

Partition the ZPFBS of length \( n \) into

- \( P_0 \): those that end in "0"
- \( P_1 \): those that end in "1"

\[ |P_0| = \# \text{ of ZPFBS of length } n-2 = zp(n-2) \ (\text{by } H(n)) \]

\[ |P_1| = \# \text{ of ZPFBS of length } n-1 = zp(n-1) \ (\text{by } H(n)) \]
What is \( zpfbs(n) \)?

use the complete induction outline

So the # of ZPFBS of length \( n \) is

\[
|P_{0}| + |P_{1}| = zp(n-2) + zp(n-1) = zp(n)
\]

as claimed.

Verify base cases:

\[
\begin{align*}
zpfs(1) & \rightarrow ("0", "1") \rightarrow 2 = zp(1) \\
zpfs(0) & \rightarrow ("") \rightarrow 1 = zp(0)
\end{align*}
\]

So the claim holds \( \forall n \in \mathbb{N} \).
Base Cases in Complete Induction

Previously, in Simple Ind:
base case is anything not
covered in Inductive Step

Complete Induction:
still anything not covered in
Inductive Step but can
have no explicit base case.
Every natural number greater than 1 has a Prime Factorization

Reminders

- Integer $m$ is divisible by integer $n$ (or $n$ is a divisor of $m$) if $\frac{m}{n}$ is an integer, i.e., $\exists k \in \mathbb{N}, m = k \times n$.

- Integer $n$ is prime if $n \geq 2$ and $n$ has no divisor in the set $\{2, \ldots, n - 1\}$ (i.e., $n$’s only divisors are 1 and $n$).

- A prime factorization of an integer $n$ is a sequence of prime numbers whose product equals $n$. e.g., $84 = 2 \times 2 \times 3 \times 7$ – because it’s a sequence, it can contain the same number multiple times.
Every natural number greater than 1 has a prime factorization. Use the complete induction outline.

**Inductive Step:** Let $n \in \mathbb{N}, n \geq 2$

Assume $H(n) : \forall i \in \mathbb{N}, 2 \leq i < n$

i has a P.F.

Show $H(n) \rightarrow C(n) : n$ has a P.F.

Case: $n$ is prime, then $n$ is its own P.F.

Case: $n$ is composite

then $n$ has a divisor $a$ s.t. $a \geq 2$

and $a < n$. 
After a certain natural number \( n \), every postage can be made up by combining 3- and 5-cent stamps.

What is the “certain natural number”?

That is, \( n = a \cdot b \), \( b \geq 2 \) and \( b < n \).

By \( H(n) \), \( a \) and \( b \) have P.F.

\[
a = p_1 \times \ldots \times p_j
\]

where \( p \)'s and \( q \)'s are prime

\[
b = q_1 \times \ldots \times q_k
\]

Then \( n = p_1 \times \ldots \times p_j \times q_1 \times \ldots \times q_k \), a P.F.

So \( C(n) \) follows from \( H(n) \). \( \Rightarrow \) \( n \) has P.F.

That's it, no explicit base case b/c the inductive step covers everything.
After a certain natural number $n$, every postage can be made up by combining 3- and 5-cent stamps.

Use the complete induction outline:

What is the "certain number"? $\Rightarrow$ experimentation

3 ✓
4 x
5 ✓
6 ✓ (2x3)
7 x
8 ✓ 3 + 5
9 ✓ 3x3
10 ✓ 2x5
11 ✓ 2x3 + 5

**Conjecture:** $\forall n \in \mathbb{N}, n \geq 8,
n$ can be made by combining 3's and 5's.

**Inductive Step:** Let $n \in \mathbb{N}, n \geq 8$.

Assume $H(n)$: $\forall i \in \mathbb{N}, 8 \leq i < n$, postage of $i\notin$ can be made with 3¢ and 5¢ stamps.

Show $H(n) \Rightarrow C(n)$: $n\notin$ can be made 3¢ + 5¢

(Intuition $\Rightarrow$ if we know $m - 3$ can be made
with 3¢ + 5¢, then we can get $m$ by
adding one more 3¢)
After a certain natural number \( n \), every postage can be made up by combining 3- and 5-cent stamps use the complete induction outline

Case: \( n = 8 \) : \( 1 \times 3\, \text{c} + 1 \times 5\, \text{c} \)

Case: \( n = 9 \) : \( 3 \times 3\, \text{c} \)

Case: \( n = 10 \) : \( 2 \times 5\, \text{c} \)

Case: \( n \geq 11 \)

Since \( n - 3 \geq 8 \), and \( n - 3 < n \), by \( H(n) \) we know that postage of \( (n-3)\, \text{c} \) can be made with \( 3\, \text{c} + 5\, \text{c} \).

Adding one more \( 3\, \text{c} \) will give postage for \( n\, \text{c} \).

Conclude: \( n \in \mathbb{N}, n \geq 8 \), postage of \( n\, \text{c} \) can be made with \( 3\, \text{c} + 5\, \text{c} \) stamps