The aim of this assignment is to give you some practice with various forms of induction. For each question below you will present a proof by induction. For full marks you will need to make it clear to the reader that the base case(s) is/are verified, that the inductive step follows for each element of the domain (typically the natural numbers), where the inductive hypothesis is used and that it is used in a valid case.

Your assignment must be typed to produce a PDF document (hand-written submissions are not acceptable). You may work on the assignment in groups of 1, 2, or 3, and submit a single assignment for the entire group on MarkUs.

1. Consider the Fibonacci-esque function \( g \):

\[
g(n) = \begin{cases} 
1, & \text{if } n = 0 \\
3, & \text{if } n = 1 \\
g(n - 2) + g(n - 1) & \text{if } n > 1
\end{cases}
\]

Use complete induction to prove that if \( n \) is a natural number greater than 1, then \( 2^{n/2} \leq g(n) \leq 2^n \). You may not derive or use a closed-form for \( g(n) \) in your proof.

2. Suppose \( B \) is a set of binary strings of length \( n \), where \( n \) is positive (greater than 0), and no two strings in \( B \) differ in fewer than 2 positions. Use simple induction to prove that \( B \) has no more than \( 2^n - 1 \) elements.

3. Define \( T \) as the smallest set of strings such that:
   
   (a) "b" \( \in T \)
   
   (b) If \( t_1, t_2 \in T \), then \( t_1 + "ene" + t_2 \in T \), where the + operator is string concatenation.

Use structural induction to prove that if \( t \in T \) has \( n \) "b" characters, then \( t \) has \( 2n - 2n \) "e" characters.

4. On page 79 of the Course Notes the quantity \( \phi = (1 + \sqrt{5})/2 \) is shown to be closely related to the Fibonacci function. You may assume that \( 1.61803 < \phi < 1.61804 \). Complete the steps below to show that \( \phi \) is irrational.

   (a) Show that \( \phi(\phi - 1) = 1 \).

   (b) Rewrite the equation in the previous step so that you have \( \phi \) on the left-hand side, and on the right-hand side a fraction whose numerator and denominator are expressions that may only have integers, + or -, and \( \phi \). There are two different fractions, corresponding to the two different factors in the original equation's left-hand side. Keep both fractions around for future consideration.
(c) Assume, for a moment, that there are natural numbers $m$ and $n$ such that $\phi = n/m$. Re-write the right-hand side of both equations in the previous step so that you end up with fractions whose numerators and denominators are expressions that may only have integers, $+$ or $-$, $m$ and $n$.

(d) Use your assumption from the previous part to construct a non-empty subset of the natural numbers that contains $m$. Use the Principle of Well-Ordering, plus one of the two expressions for $\phi$ from the previous step to derive a contradiction.

(e) Combine your assumption and contradiction from the previous step into a proof that $\phi$ cannot be the ratio of two natural numbers. Extend this to a proof that $\phi$ is irrational.

5. Consider the function $f$, where $3 \div 2 = 1$ (integer division, like $3//2$ in Python):

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ f^2(n \div 3) + 3f(n \div 3) & \text{if } n > 0 \end{cases}$$

Use complete induction to prove that for every natural number $n$ greater than 2, $f(n)$ is a multiple of 7. NB: Think carefully about which natural numbers you are justified in using the inductive hypothesis for.