Floating Point

CSC207 Fall 2017

Computer Science
UNIVERSITY OF TORONTO

Ariane 5 Rocket Launch

Ariane 5 rocket explosion

- In 1996, the European Space Agency’s Ariane 5 rocket exploded 40 seconds after launch.
- During conversion of a 64-bit to a 16-bit format, overflow occurred: the number was too big to store in 16 bits.
- This hadn’t been expected because the data (acceleration reported by sensors) had never been this large before. But this new rocket was faster than its predecessor.
- $7 billion of R&D had been invested in this rocket.
- Reference: http://www.around.com/ariane.html

Example 1

- Perform some simple arithmetic, and check that the laws of mathematics hold.
  - Code: `Adding.java`

Is Java broken?

It's not only Java. Check this out in Python:

```
>>> x = 0.1
>>> sum = x + x + x
>>> sum == 0.3
False
>>> sum
0.30000000000000004
>>> bigger = 1.0
>>> s = 1.0e-6
>>> sum1 = s + s + s + s + s + s + s + s + s + s + bigger
>>> sum2 = bigger + s + s + s + s + s + s + s + s + s + s
>>> sum1 == sum2
False
>>> sum1
1.00001
>>> sum2
1.0000099999999992
```

Representing numbers

- It all makes sense if you understand how "real" numbers are represented.
- First, consider an `int` like `42`. Hardware doesn't directly represent 4s or 2s — everything is binary.
- 42 =

  $$\mathbf{1} \times 2^5 + \mathbf{0} \times 2^4 + \mathbf{1} \times 2^3 + \mathbf{0} \times 2^2 + \mathbf{1} \times 2^1 + \mathbf{0} \times 2^0$$
- So 42 can be represented by 101010 (base 2).

Representing fractions

- Fractions can be handled using the same approach.
- Example: 0.4375 =

  0 x 2^{-1} + **1** x 2^{-2} + **1** x 2^{-3} + **1** x 2^{-4}

  = 0/2 + 1/4 + 1/8 + 1/16

  = 0.25 + 0.125 + 0.0625

  = 0.4375
- So we can represent 0.4375 using 0.0111 (base 2).
- Another example: 0.1 =

  0.0001100110011001100110011001100...

  - 0.1 does not have a finite binary representation

Some problem numbers

- You already know from math that some numbers do not have a finite representation.
- Even worse, some numbers that have a finite representation in decimal do not in binary!
  - (Thought question: is the reverse true?)
- Computer systems have finite memory.
  But we need to represent numbers that take an infinite number of bits.
- Solution?

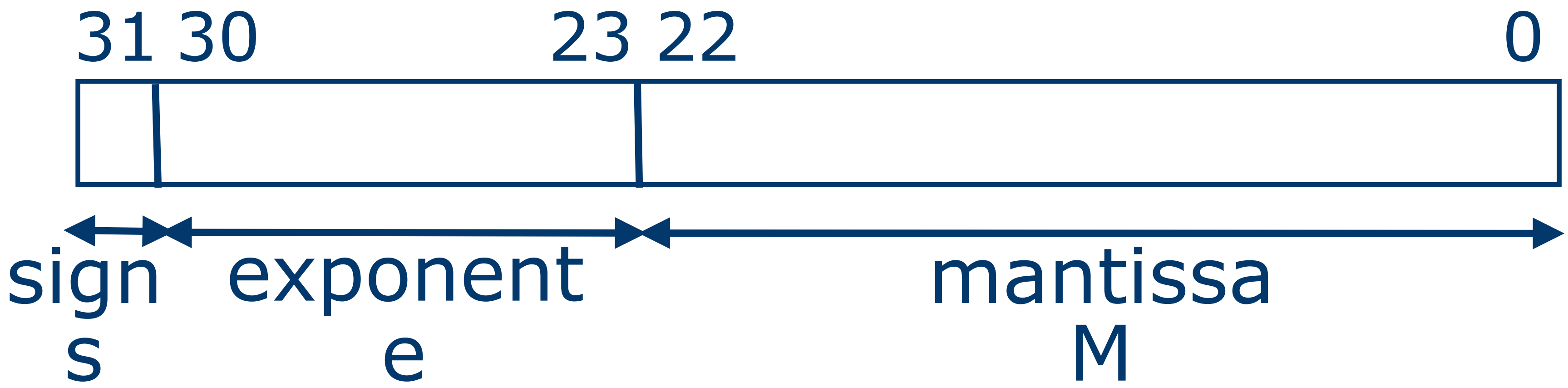
IEEE-754 Floating Point

- Like a binary version of scientific notation
- 32 bits for a float (64 bits for a double) as follows:
  - 1 bit for the **sign**
  - 8 bits for the **exponent** e
  - 23 bits for the **mantissa** (significand) M

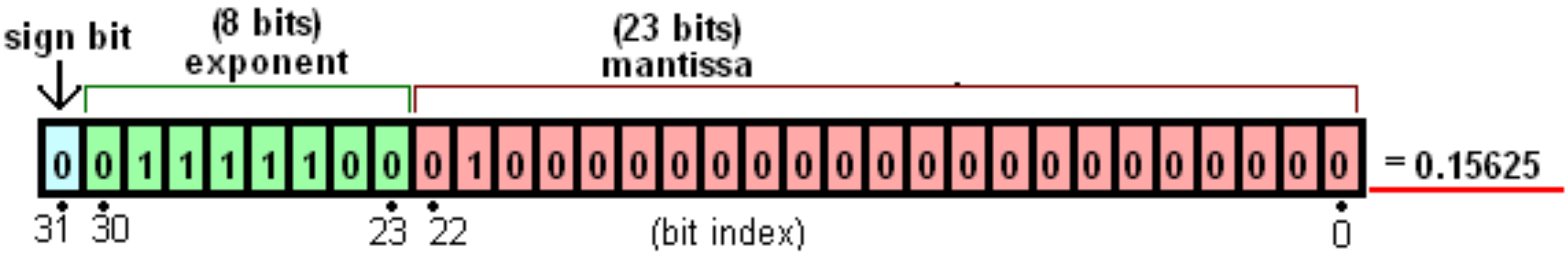
Allocation of 32 bits

- **1 bit** for the **sign**: 1 for negative and 0 for positive
- **8 bits** for the **exponent** e
  - To allow for negative exponents, 127 is added to that exponent to get the representation.
    We say that the exponent is "biased" by 127.
  - So the range of possible exponents is
    not 0 to 2^8-1 = 0 to 255,
    but (0-127) to (255-127) = -127 to 128.
- **23 bits** for the **mantissa** M
  - Since the first bit must be 1, we don't waste space storing it!

IEEE-754 Floating Point



$$(-1)^{s} * (1 + M) * 2^{e-127}$$



$$(-1)^{\text{sign}} \left(1 + \sum_{i=1}^{23} b_{23-i} 2^{-i}\right) \times 2^{(e-127)}$$

- $\text{sign} = 0$
- $1 + \sum_{i=1}^{23} b_{23-i} 2^{-i} = 1 + 2^{-2} = 1.25$
- $2^{(e-127)} = 2^{124-127} = 2^{-3}$

thus:

- $\text{value} = 1.25 \times 2^{-3} = 0.15625$

Rounding

- If we have to lose some digits, we don't just truncate, we round.
- In rounding a decimal to a whole number, an issue arises: If we have a 0.5, do we round up or down?
- If we always round up, we are biasing towards higher values.
- "Proper" rounding: round to the nearest even number.
  E.g., 17.5 is rounded up to 18 but 16.5 is rounded down to 16.
- The IEEE standard uses proper rounding also.

Historical aside

- 30 years ago, computer manufacturers each had their own standard for floating point.
- Problem? Writing portable software!
- Advantage to manufacturers? Customers got locked in to their particular computers.
- In the late 1980s, the IEEE produced the standard that now virtually all follow.
- William Kahan spearheaded the effort, and won the 1989 Turing Award for it.

Back to the example (`Adding.java`)

- As we saw, 0.1 cannot be represented exactly in binary, leading to the unexpected result.
- And adding a very small quantity to a very large quantity can mean the smaller quantity falls off the end of the mantissa.
- But if we add small quantities to each other, this doesn’t happen.
  And if they accumulate into a larger quantity, they may not be lost when we finally add the big quantity in.

Examples 2 and 3

- This seems contrived, but consider some value that accumulates in a loop.
  - Code: `Totalling.java`
- Or consider adding up a list of doubles, what should you do?
  - Code: `ArrayTotal.java`

Lessons

- When adding floating point numbers, add the smallest first.
- More generally, try to avoid adding dissimilar quantities.
- Specific scenario: When adding a list of floating point numbers, sort them first.

Example 4

- Repeat a task for values in a particular range with an increment of 0.1.
  - For example, for values 0.1 to 0.5 with an increment of 0.1.
  - For example, for values 1.1 to 1.5 with an increment of 0.1.
  - Code: `FunctionValues.java`

Lessons

- Don’t use floating point variables to control what is essentially a counted loop.

- Also: Notice that we wrote

  ```
  x = 1.0 + i * 0.1;
  ```

  instead of initializing `x` to `1.0` and then repeatedly adding `0.1`.
  Why? Fewer total arithmetic operations means fewer rounding errors are introduced.

- Use fewer arithmetic operations where possible.

Example 5

- A very simple program that just prints the same variable using different formats.
  - Code: `Examine.java`

What happened?

- We shouldn't be surprised by now to find out that 4/5 can't be represented exactly in a float. Lots of things can't.
- But the represented value should be off by a tiny bit. What are all these extra digits??

bit 24

- 4/5 = 1.10011001100110011001100**0**11001100 ... x 2^{-1}
- It gets rounded to

  1.1001100110011001100110**1** x 2^{-1}

- When we ask to print it as a decimal number, it gets converted.
  The exact equivalent is

  0.800000011920928955078125000000

- But only 7 of those digits are significant.

Lesson

- Don’t print more precision in your output than you are holding.

Why does this matter?

Patriot missile accident

- In 1991, an American missile failed to track and destroy an incoming missile. Instead it hit a US Army barracks, killing 28.
- The system tracked time in tenths of seconds. The error in approximating 0.1 with 24 bits was magnified in its calculations.
- At the time of the accident, the error corresponded to 0.34 seconds. A Patriot missile travels about half a km in that time.
- Reference: http://www.ima.umn.edu/~arnold/disasters/patriot.html

Sinking of an oil rig

- In 1992, the Sleipner A oil and gas platform sank in the North Sea near Norway.
- Numerical issues in modelling the structure caused shear stresses to be underestimated by 47%.
- As a result, concrete walls were not built thick enough.
- Cost: $700 million
- Reference: http://www.ima.umn.edu/~arnold/disasters/sleipner.html

What should you do?

"95% of folks out there are completely clueless about floating-point."

James Gosling

Follow the lessons

- Use double instead of float.
- When adding floating point numbers, add the smallest first.
- More generally, try to avoid adding dissimilar quantities.
- Specific scenario: When adding a list of floating point numbers, sort them first.
- Don't use floating point variables to control what is essentially a counted loop.
- Use fewer arithmetic operations where possible.
- Don't print more precision in your output than you are holding.

CSC336 Numerical Methods

- The study of computational methods for solving problems in linear algebra, non-linear equations, and approximation. The aim is to give students a basic understanding of both floating-point arithmetic and the implementation of algorithms used to solve numerical problems, as well as a familiarity with current numerical computing environments.