# Homework Assignment 2

CSC 200Y: Social and Economic Networks

Out: November 9, 2015
Due: November 25, 2015

**Be sure to include your name, student number and tutorial room with your assignment. If your handwriting is *possibly* illegible, be sure to hand in your assignment in some typed form.**

1. Consider the following game in matrix form with two players. Payoffs for the row player Peter are indicated first in each cell, and payoffs for the column player Ravin are second.

| | $C$ | $D$ |
|---|---|---|
| $A$ | $10, 16$ | $14, 24$ |
| $B$ | $15, 18$ | $6, 10$ |

(a) Does either player have a dominant strategy? Explain your answer.

(b) What are the *pure* strategy Nash equilibria of this game? Justify your answer. If there is more than one pure equilibria, which would Ravin prefer? What is the Price of Anarchy (with respect to pure NE) for this game?

(c) This game has a *strictly mixed* strategy Nash equilibria in which both Peter and Ravin play each of their actions with positive probability. What are the mixed strategies for each player in this equilibrium? Show how you would compute such a mixed equilibirum and verify that your mixed strategies are indeed in equilibrium.

(d) Suppose that Peter and Ravin play this game repeatedly once per day, each time choosing their actions according to some strategy. Peter claims that he will play his mixed strategy accoording to the probabilities you calculated in part (c). Ravin decides to take Peter at his word and after a few days commits to play a pure stratgey from now on. Does it matter which one he plays and if so which one will he play?

(e) After Ravin commits to play a pure strategy as in part (d) , should Peter reneg on his word and play a different strategy knowing that Ravin has committed to a pure strategy and will never change? Why or why not?

2. Consider the following game involving two players, Anni and Barak. Anni has two actions (or strategies), $S$ and $T$, corresponding to rows in the game matrix below. Barak has three actions $X$, $Y$, and $Z$, corresponding to columns. Payoffs in each matrix cell list the row player (Anni) payoff first, and the column player (Barak) second. So, for example, in the strategy profile $(S, X)$, Anni's payoff is 3 while Barak's is 4.

| | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| $S$ | $3, 4$ | $4, 6$ | $2, 5$ |
| $T$ | $2, 8$ | $1, 2$ | $8, 5$ |

(a) If Anni plays $T$, what is Barak's best response?

(b) If Barak plays $Z$, what is Anni's best response?

(c) This game has one pure strategy Nash equilibrium: what is it?

(d) Is the pure strategy Nash equilibrium in part (c) socially optimal? What is the Price of Anarchy of this game? Justify your response.

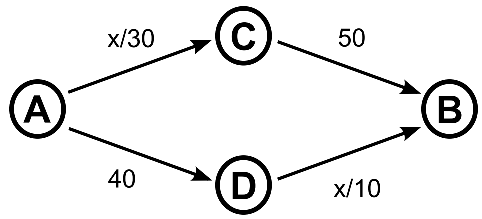
(e) Consider the following *mixed* strategy profile:

- Anni plays $S$ with probability $\frac{3}{4}$, and $T$ with probability $\frac{1}{4}$.
- Barak plays $X$ with probability $\frac{3}{5}$, $Y$ with probability $\frac{1}{5}$, and $Z$ with probability $\frac{1}{5}$.

Explain why this is a mixed Nash equilibrium of the game.

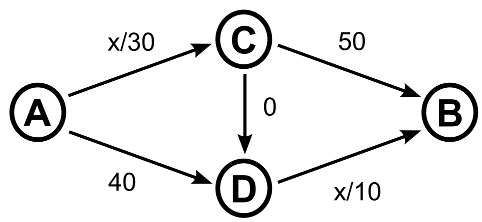
3. The following question refers to the two traffic networks illustrated below. In each case, 900 cars need to travel from city $A$ to city $B$ each day, and must pass through city $C$ or city $D$ en route ($C$ and $D$ are separated by a river). Driving times are indicated on each link, and sometimes depend on the number of cars $x$ that travel on the link. So, for example, if all 900 cars took route $A$–$C$–$B$ (in either of the networks below), the commute time would be $\frac{900}{30} + 50 = 80$ minutes for each car.

(a) The original network consists of two possible routes from $A$ to $B$ as shown:




What is the equilibrium traffic flow in this network (i.e., how many cars take each of the two routes in equilibrium)? What is the average driving time on each route? Justify your answer.

(b) A new one-way bridge is added that allows a direct, nearly instantaneous connection from $C$ to $D$. For our purposes, we consider its travel time to be 0, as shown in this new network:




Will any cars now travel on the link $A$–$D$ to get to $B$? Why or why not?

(c) What is the equilibrium traffic flow in this new network (as described in (b)), i.e., how many cars travel on each route in equilibrium? What happens to the average driving time? Justify your answer.

4. Suppose two Yoga instructors are offering small personal classes to student groups of size 1, 2 or 3. Each instructor teaches a different style of yoga. The Ashtanga instructor $A$ charges \$10 for one student; but she offers a small discount, charging \$8 per student, if 2 or 3 students enrol. The Bikram instructor $B$ charges \$14 for one student; but he also offers discounted per-student prices of \$12 for two students and \$8 for three students. The prices and discounts are summarized as follows:

| Number of Students | 1 | 2 | 3 |
|---|---|---|---|
| A's Per-Student Price | 10 | 8 | 8 |
| B's Per-Student Price | 14 | 12 | 8 |

Three friends Xin, Yael, and Zach are each considering taking one of the classes. But they each have different preferences or *valuations* for the classes. If they take a class that has valuation $v$ and charges them a price $p$, then their utility or net *payoff* for that class is $v - p$. Their valuations are as follows:

| | Xin | Yael | Zack |
|---|---|---|---|
| Ashtanga | 17 | 16 | 13 |
| Bikram | 20 | 18 | 20 |

Putting these together, we see that if Xin takes the Ashtanga class by himself, his net payoff is $17 - 10 = 7$, since his valuation is 17 and he pays the undiscounted price of 10. If one (or both) of his friends took the class with him, his payoff would be 9 (since he would receive the discounted price of 8). Similarly for other students and combinations.

(a) Suppose all three friends must each sign up for one of the classes, simultaneously, without knowledge of what classes their friends are joining. They do know each other's valuations and the prices and discounts available.

(i) Formulate this problem as a matrix game with three players $X, Y, Z$, with two moves each $A$ and $B$ corresponding to their choice of class. The *payoffs* in the matrix are their net payoffs based on their valuation for the class they join and the (possibly discounted) price they pay. Recall our notation from class for three-player games (see also Exercise 13, Ch.6 of the text): describe the game by specifying *two* matrices, one corresponding to $Z$ selecting $A$, the other $Z$ selecting $B$; each matrix is $2 \times 2$, representing the two moves of $X$ and $Y$; and each entry in the matrix contains the payoffs for all three players. Write the costs in each cell of the matrix in "player order" $X, Y, Z$.

(ii) Which players (if any) have a dominant strategy in this game? Explain.

(iii) What are the pure Nash equilibria of this game? Are any of the pure Nash equilibria pareto optimal?

(iv) What strategy profile has the highest social welfare (and is it a Nash equilibrium)?

Continued on next page

(b) Now suppose that the friends each sign up for classes *in sequence*. Once one person signs up for a class, they tweet their choice, so that the remaining friends are informed of this choice before they make their own choices.

(i) Draw the extensive form game tree for this game, assuming that players announce their choices in the following order: first Xin chooses a class, then Yael chooses, then Zach chooses.

(ii) Provide a description of the *subgame perfect equilibrium* in this extensive form game. (This is the equilibrium that is supported by backward induction.) State what actions would be chosen by each player at each stage of the game and what their payoffs will be. Does this correspond to any of the Nash equilibria of the matrix form game?

(iii) Would the the outcome of the game change if the order of the players' choices were reversed? Why or why not?

(iv) Suppose that we changed Zach's valuations as follows: Ashtanga now is worth $15$ and Bikram is worth $14$. Rewrite the extensive form game with these new payoffs (note: only Zach's payoffs will change at each leaf of the tree). Describe an equilibrium in this new game that is *not* a subgame perfect equilibrium. Justify your answer. *(Hint: consider how $Z$ might "threaten" $X$ and $Y$ to get a better payoff for himself)*

5. Suppose an autioneer announces a second-price, sealed-bid auction for a famous painting. There are three potential buyers with independent, private values. Tom values the painting at \$2M, Justin at \$4M and Stephen at \$8M. Each buyer will play a dominant strategy if they can get to the auction.

   - If all three arrive at the auction, what is the social welfare of the auction and how much revenue will the auctioneer obtain?
   - Suppose that the auctioneer is in a hurry and decides to run the auction for the first two people who show up and does not let the last person participate. Suppose that each person is the last person to show up with with equal probability (i.e. probability $\frac{1}{3}$). What is the expected social welfare and what is the expected revenue?