Today’s Topics

- **Review: Proof Structures for Quantifiers, Implications and Conjunctions**

- **Proof Structure for Disjunction**

- **Proof by Cases**
Chapter 3

Formal Proofs

Review: Proof Structures for Quantifiers, Implications and Conjunctions
Proof of Multiple Quantifiers

Structure

- Prove $\forall x \in D, \exists y \in E, P(x, y)$
  - Assume $x \in D$.  # $x$ is a typical element of $D$
    - Let $y = \_\_\_$.  # choose a particular element of the domain
    - Then $y \in E$.  # this may be obvious, otherwise prove it
      - # prove $P(x, y)$
      - Then $P(x, y)$.
    - Then $\exists y \in E, P(x, y)$.  # introduce existential
      - Then $\forall x \in D, \exists y \in E, P(x, y)$.  # introduce universal
Proof of Multiple Quantifiers

Structure

- Prove $\exists x \in D, \forall y \in E, P(x, y)$
  
  Let $x = \ldots$.  # choose a particular element of the domain
  Then $x \in D$.  # this may be obvious, otherwise prove it
  Assume $y \in E$.  # $y$ is a typical element of $E$
  
  $\vdash$  # prove $P(x, y)$
  Then $P(x, y)$.
  Then $\forall x \in D, P(x, y)$.  # introduce universal
  Then $\exists y \in E, \forall x \in D, P(x, y)$.  # introduce existential
Proof of Conjunction

Structure

- Prove $\forall x \in D, P(x) \land Q(x)$
  
  Assume $x \in D$.  # $x$ is a typical element of $D$
  
  Then $P(x)$.
  
  Then $Q(x)$.
  
  Then $P(x) \land Q(x)$.  # introduce conjunction
  
  Then $\forall x \in D, P(x) \land Q(x)$.  # introduce universal
Chapter 3

Formal Proofs

Proof Structure for Disjunction
Proof of Disjunction

Structure

- Prove $\forall x \in D, P(x) \lor Q(x)$
- Assume $x \in D$.  # $x$ is a typical element of $D$
  \begin{itemize}
  \item # prove $P(x)$
  \begin{itemize}
  \item Then $P(x)$.
  \item Then $P(x) \lor Q(x)$.  # introduce disjunction
  \end{itemize}
  \end{itemize}
  Then $\forall x \in D, P(x) \lor Q(x)$.  # introduce universal
- Assume $x \in D$.  # $x$ is a typical element of $D$
  \begin{itemize}
  \item # prove $Q(x)$
  \begin{itemize}
  \item Then $Q(x)$.
  \item Then $P(x) \lor Q(x)$.  # introduce disjunction
  \end{itemize}
  \end{itemize}
  Then $\forall x \in D, P(x) \lor Q(x)$.  # introduce universal
Chapter 3

Formal Proofs

Proof by Cases
Proof by Cases

Implications with Disjunctive Antecedents

- Consider an **implication** which has a **disjunction** as the **antecedent**:
  - \( S_1 : (A_1 \lor A_2) \Rightarrow C \).
- How can we prove \( S_1 \)?
  - \((A_1 \lor A_2) \Rightarrow C\) is equivalent with \((A_1 \Rightarrow C) \land (A_2 \Rightarrow C)\).

General Structure

Assume \( A_1 \lor A_2 \).

**Case 1**: Assume \( A_1 \).

- \[
  \begin{align*}
  & : \quad \text{# prove } C \\
  & \text{Then } C.
  \end{align*}
\]

Then \( A_1 \Rightarrow C \).  \# assuming \( A_1 \) leads to \( C \)

**Case 2**: Assume \( A_2 \).

- \[
  \begin{align*}
  & : \quad \text{# prove } C \\
  & \text{Then } C.
  \end{align*}
\]

Then \( A_2 \Rightarrow C \).  \# assuming \( A_2 \) leads to \( C \)

Then \((A_1 \Rightarrow C) \land (A_2 \Rightarrow C)\).  \# introduce conjunction

Then \((A_1 \lor A_2) \Rightarrow C\).  \# logically equiv. the previous statement
Proof by Cases

### General Case

- **S₂**: \((A_1 \lor \ldots \lor A_n) \Rightarrow C\).
- **S₂** is equivalent with \((A_1 \Rightarrow C) \land \ldots \land (A_n \Rightarrow C)\).

### General Structure

Assume \(A_1 \lor \ldots \lor A_n\).

**Case 1**: Assume \(A_1\).

\[\vdash \# \text{ prove } C\]
Then \(C\).

Then \(A_1 \Rightarrow C\). \# assuming \(A_1\) leads to \(C\)

\[\vdash \]

**Case n**: Assume \(A_n\).

\[\vdash \# \text{ prove } C\]
Then \(C\).

Then \(A_n \Rightarrow C\). \# assuming \(A_n\) leads to \(C\)
Then \((A_1 \Rightarrow C) \land \ldots \land (A_n \Rightarrow C)\). \# introduce conjunction
Then \((A_1 \lor \ldots \lor A_n) \Rightarrow C\). \# logically equiv. the previous statement
Proof by Cases

**General Case**

- **Assumption:** \((A_1 \lor \ldots \lor A_n)\).
- **Claim:** \(C\).

**General Structure**

Assume \(A_1 \lor \ldots \lor A_n\).

**Case 1:** Assume \(A_1\).

\[\vdash \text{ # prove } C\]
Then \(C\).

\[\vdots\]

**Case n:** Assume \(A_n\).

\[\vdash \text{ # prove } C\]
Then \(C\).

Then \(C\). # assuming \(A_1 \lor \ldots \lor A_n\) leads to \(C\)
Proof by Cases

Exercise

- Prove that if \( n \) is an integer number, then \( n^2 + n \) is even.

Solution

- **Step 1:** Translate the claim to logical notation.
  - For all integers \( n \), \( n^2 + n \) is even.
    \[ \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k. \]

- **Step 2:** Find a plan for the proof:
  - Consider two cases: \( n \) is odd or \( n \) is even.

- **Step 3:** Translate the assumptions/facts to logical notation
  - \[ \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k). \]

- **Step 4:** Choose an appropriate proof structure. Use the assumptions/facts to prove the claim.
Proof by Cases

**Exercise**

- **Assumption:** \( \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k). \)
- **Claim:** \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k. \)

**Solution**

Assume \( n \in \mathbb{Z}. \) \( \# n \) is a typical integer number

Then \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k. \) \( \# \) introduction of universal
Proof by Cases

Exercise

- **Assumption**: \( \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k) \).
- **Claim**: \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \).

Solution

Assume \( n \in \mathbb{Z} \). \# \( n \) is a typical integer number

\[ \vdots \]

Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \).
Then \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# introduction of universal
Proof by Cases

Exercise

- **Assumption:** \( \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k) \).
- **Claim:** \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \).

Solution

Assume \( n \in \mathbb{Z} \). # \( n \) is a typical integer number

Then \( (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k) \). # by Assumption, \( n \in \mathbb{Z} \)

Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \).

Then \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \). # introduction of universal
Proof by Cases

Exercise

- **Assumption:** $\forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k)$.
- **Claim:** $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k$.

Solution

Assume $n \in \mathbb{Z}$. \# $n$ is a typical integer number

Then $(\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k)$. \# by Assumption, $n \in \mathbb{Z}$

Case 1: Assume $\exists k \in \mathbb{Z}, n = 2k + 1$.

\begin{itemize}
  \item Then $\exists k \in \mathbb{Z}, n^2 + n = 2k$. \# true in all (both) possible cases

\end{itemize}

Then $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k$. \# introduction of universal
Proof by Cases

Exercise

- **Assumption:** \( \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k) \).
- **Claim:** \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \).

Solution

Assume \( n \in \mathbb{Z} \). # \( n \) is a typical integer number

Then \((\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k)\). # by Assumption, \( n \in \mathbb{Z} \)

**Case 1:** Assume \( \exists k \in \mathbb{Z}, n = 2k + 1 \).

Let \( k_0 \in \mathbb{Z} \) be such that \( n = 2k_0 + 1 \). # instantiate existential

...#

Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \). # true in all (both) possible cases

Then \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \). # introduction of universal
Proof by Cases

Exercise

- **Assumption:** \( \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k). \)
- **Claim:** \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k. \)

Solution

Assume \( n \in \mathbb{Z}. \)  # \( n \) is a typical integer number

Then \( (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k). \)  # by Assumption, \( n \in \mathbb{Z} \)

Case 1: Assume \( \exists k \in \mathbb{Z}, n = 2k + 1. \)

Let \( k_0 \in \mathbb{Z} \) be such that \( n = 2k_0 + 1. \)  # instantiate existential

Then \( n^2 + n = n(n + 1) = (2k_0 + 1)(2k_0 + 2) = 2(2k_0 + 1)(k_0 + 1). \)  # some algebra

\[
\vdots
\]

\[
\vdots
\]

Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k. \)  # true in all (both) possible cases

Then \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k. \)  # introduction of universal
Proof by Cases

Exercise

- **Assumption:** $\forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k)$.
- **Claim:** $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k$.

Solution

Assume $n \in \mathbb{Z}$. # $n$ is a typical integer number

Then $(\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k)$. # by Assumption, $n \in \mathbb{Z}$

Case 1: Assume $\exists k \in \mathbb{Z}, n = 2k + 1$.

Let $k_0 \in \mathbb{Z}$ be such that $n = 2k_0 + 1$. # instantiate existential

Then $n^2 + n = n(n + 1) = (2k_0 + 1)(2k_0 + 2) = 2(2k_0 + 1)(k_0 + 1)$. # some algebra

Then $\exists k \in \mathbb{Z}, n^2 + n = 2k$. # $k = (2k_0 + 1)(k_0 + 1) \in \mathbb{Z}$

Then $\exists k \in \mathbb{Z}, n^2 + n = 2k$. # true in all (both) possible cases

Then $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k$. # introduction of universal
Proof by Cases

Exercise

- **Assumption:** $\forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k)$.
- **Claim:** $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k$.

Solution

Assume $n \in \mathbb{Z}$.  

# $n$ is a typical integer number

Then $(\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k)$.  

# by Assumption, $n \in \mathbb{Z}$

Case 1: Assume $\exists k \in \mathbb{Z}, n = 2k + 1$.

Let $k_0 \in \mathbb{Z}$ be such that $n = 2k_0 + 1$.  

# instantiate existential

Then $n^2 + n = n(n + 1) = (2k_0 + 1)(2k_0 + 2) = 2(2k_0 + 1)(k_0 + 1)$.  

# some algebra

Then $\exists k \in \mathbb{Z}, n^2 + n = 2k$.  

# $k = (2k_0 + 1)(k_0 + 1) \in \mathbb{Z}$

Case 2: Assume $\exists k \in \mathbb{Z}, n = 2k$.

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Then $\exists k \in \mathbb{Z}, n^2 + n = 2k$.  

# true in all (both) possible cases

Then $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k$.  

# introduction of universal
Proof by Cases

Exercise

- **Assumption:** \( \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k) \).
- **Claim:** \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \).

Solution

Assume \( n \in \mathbb{Z} \). \# \( n \) is a typical integer number

Then \((\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k)\). \# by Assumption, \( n \in \mathbb{Z} \)

**Case 1:** Assume \( \exists k \in \mathbb{Z}, n = 2k + 1 \).

Let \( k_0 \in \mathbb{Z} \) be such that \( n = 2k_0 + 1 \). \# instantiate existential

Then \( n^2 + n = n(n + 1) = (2k_0 + 1)(2k_0 + 2) = 2(2k_0 + 1)(k_0 + 1) \). \# some algebra

Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# \( k = (2k_0 + 1)(k_0 + 1) \in \mathbb{Z} \)

**Case 2:** Assume \( \exists k \in \mathbb{Z}, n = 2k \).

Let \( k_0 \in \mathbb{Z} \) be such that \( n = 2k_0 \). \# instantiate existential

Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# true in all (both) possible cases

Then \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# introduction of universal
Proof by Cases

Exercise

- **Assumption:** \( \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k) \).
- **Claim:** \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \).

Solution

Assume \( n \in \mathbb{Z} \). \# \( n \) is a typical integer number

Then \( (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k) \). \# by Assumption, \( n \in \mathbb{Z} \)

Case 1: Assume \( \exists k \in \mathbb{Z}, n = 2k + 1 \).

Let \( k_0 \in \mathbb{Z} \) be such that \( n = 2k_0 + 1 \). \# instantiate existential

Then \( n^2 + n = n(n + 1) = (2k_0 + 1)(2k_0 + 2) = 2(2k_0 + 1)(k_0 + 1) \). \# some algebra

Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# \( k = (2k_0 + 1)(k_0 + 1) \in \mathbb{Z} \)

Case 2: Assume \( \exists k \in \mathbb{Z}, n = 2k \).

Let \( k_0 \in \mathbb{Z} \) be such that \( n = 2k_0 \). \# instantiate existential

Then \( n^2 + n = n(n + 1) = 2k_0(2k_0 + 1) = 2[k_0(2k_0 + 1)] \).

Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# true in all (both) possible cases

Then \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# introduction of universal
Proof by Cases

**Exercise**

- **Assumption:** \( \forall n \in \mathbb{Z}, (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k) \).
- **Claim:** \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \).

**Solution**

Assume \( n \in \mathbb{Z} \). \# \( n \) is a typical integer number
Then \( (\exists k \in \mathbb{Z}, n = 2k + 1) \lor (\exists k \in \mathbb{Z}, n = 2k) \). \# by Assumption, \( n \in \mathbb{Z} \)
Case 1: Assume \( \exists k \in \mathbb{Z}, n = 2k + 1 \).
Let \( k_0 \in \mathbb{Z} \) be such that \( n = 2k_0 + 1 \). \# instantiate existential
Then \( n^2 + n = n(n + 1) = (2k_0 + 1)(2k_0 + 2) = 2(2k_0 + 1)(k_0 + 1) \). \# some algebra
Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# \( k = (2k_0 + 1)(k_0 + 1) \in \mathbb{Z} \)
Case 2: Assume \( \exists k \in \mathbb{Z}, n = 2k \).
Let \( k_0 \in \mathbb{Z} \) be such that \( n = 2k_0 \). \# instantiate existential
Then \( n^2 + n = n(n + 1) = 2k_0(2k_0 + 1) = 2[k_0(2k_0 + 1)] \).
Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# \( k = k_0(2k_0 + 1) \in \mathbb{Z} \)
Then \( \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# true in all (both) possible cases
Then \( \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, n^2 + n = 2k \). \# introduction of universal
Proof by Cases

Exercise

Prove that the square of a natural is a multiple of 3 or a multiple of 3 plus 1.

Solution

Step 1: Translate the claim to logical notation.
- \( \forall n \in \mathbb{N}, (\exists k \in \mathbb{N}, n^2 = 3k) \lor (\exists k \in \mathbb{N}, n^2 = 3k + 1) \).

Step 2: Find a plan for the proof:
- Consider three cases: \( n = 3k \) or \( n = 3k + 1 \) or \( n = 3k + 2 \).

Step 3: Translate the assumptions/facts to logical notation
- \( \forall n \in \mathbb{N}, (\exists k \in \mathbb{N}, n = 3k \lor n = 3k + 1 \lor n = 3k + 2) \).

Step 4: Choose an appropriate proof structure. Use the assumptions/facts to prove the claim.
Proof by Cases

Structure

- Disjunction in the assumptions $\rightarrow$ proof by cases
- Disjunction in the claim $\rightarrow$ proof structure for disjunction

Assumption: $P \lor Q$.

Claim: $S \lor R$.

Assume $P \lor Q$

Case 1: Assume $P$.

\[ \therefore \quad \# \text{ prove } R \]

Then $R$.

Case 2: Assume $Q$.

\[ \therefore \quad \# \text{ prove } S \]

Then $S$.

Thus $R \lor S$. $\#$ introduce disjunction
Proof by Cases

Structure

- Disjunction in the **assumptions** $\rightarrow$ proof by cases
- Disjunction in the **claim** $\rightarrow$ proof structure for disjunction

**Assumption:** $P \lor Q$.

**Claim:** $S \lor R$.

Assume $P \lor Q$

**Case 1:** Assume $P$.

\[ \vdash \text{# prove } R \]

Then $R$.

Then $R \lor S$.  \# introduce disjunction

**Case 2:** Assume $Q$.

\[ \vdash \text{# prove } S \]

Then $S$.

Then $R \lor S$.  \# introduce disjunction

Thus $R \lor S$.  \# introduce disjunction