Chapter 2

Logical Notation

Bahar Aameri
Department of Computer Science
University of Toronto

Jan 09, 2015
Announcements

- **Tutorials:**
  - Locations and times are posted on the course web page.
  - Tutorial exercises will be posted on the course web page before Monday. Work on the exercise before the tutorial.
  - Each quiz covers all topics that you have learned during the week prior to the quiz.

- **Office hours:** Friday 12:30-1:30pm and 3:30-5pm in BA4261.
Today’s Topics

- Evaluating Quantified Statements
- Visualization with Venn Diagram
- Logical Sentences and Statements
- Negation, Conjunction, Disjunction
Chapter 2
Logical Notation

Evaluating Quantified Statements
Review: Sets

Properties and Relationships as Sets

- To describe a domain, we write **statements** that specify **properties** of objects within the domain and their **relationships**.
  
  One way of writing statements in **symbolic** notation is to treat **properties** and **relationships** as **sets**.

Example

<table>
<thead>
<tr>
<th>Emp.</th>
<th>Gender</th>
<th>Supervisor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>male</td>
<td>-</td>
</tr>
<tr>
<td>Betty</td>
<td>female</td>
<td>Doug</td>
</tr>
<tr>
<td>Carlos</td>
<td>male</td>
<td>Ellen</td>
</tr>
<tr>
<td>Doug</td>
<td>male</td>
<td>Ellen</td>
</tr>
<tr>
<td>Ellen</td>
<td>female</td>
<td>Al</td>
</tr>
<tr>
<td>Flo</td>
<td>female</td>
<td>Ellen</td>
</tr>
</tbody>
</table>

- **Property:**
  
  \[ M = \{ x \mid x \text{ is male}\}. \]
  
  \[ M = \{ Al, Carlos, Doug\}. \]

- **Relationship:**
  
  \[ S = \{ \langle x, y \rangle \mid x \text{ supervises } y\}. \]
  
  \[ S = \{ \langle Al, Ellen\rangle, \langle Ellen, Carlos\rangle, \langle Ellen, Doug\rangle, \langle Ellen, Flo\rangle, \langle Doug, Betty\rangle\}. \]
Review: Quantified Statements

- When an statement is about all the objects in the domain, the statement is a **Universal Quantification**.
  - **Universal quantifier**: $\forall$

- Examples of universally quantified statements in English:
  - All employee makes less than $55,000.$
  - Each male employee makes more than $55,000.$

- When an statement is about existence of one or more elements of a domain with a particular property, the statement is a **Existential Quantification**.
  - **Existential quantifier**: $\exists$

- Examples of existentially quantified statements in English:
  - Some employee earns over $65,000.$
  - At least one female employee earns less than $65,000.$
Evaluating Quantified Statements

- **Prove/disprove** the following universally quantified claims.
  - Every employee makes less than $55,000.
  - Every female employee makes less than $50,000.
  - There is no male employee which makes less than $30,000.

<table>
<thead>
<tr>
<th>Employee</th>
<th>Gender</th>
<th>Salary</th>
<th>Supervisor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>male</td>
<td>$60,000</td>
<td>-</td>
</tr>
<tr>
<td>Betty</td>
<td>female</td>
<td>$500</td>
<td>Doug</td>
</tr>
<tr>
<td>Carlos</td>
<td>male</td>
<td>$40,000</td>
<td>Ellen</td>
</tr>
<tr>
<td>Doug</td>
<td>male</td>
<td>$30,000</td>
<td>Ellen</td>
</tr>
<tr>
<td>Ellen</td>
<td>female</td>
<td>$50,000</td>
<td>Al</td>
</tr>
<tr>
<td>Flo</td>
<td>female</td>
<td>$20,000</td>
<td>Ellen</td>
</tr>
</tbody>
</table>

Evaluating Universally Quantified Statements

- To **prove**, verify that all elements of the domain is an example that satisfies the quantification.
- To **disprove**, give at least one counter-example that does not satisfy the quantification.
Evaluating Quantified Statements

- **Prove/disprove** the following existentially quantified claims.
  - Some employee earns less than $57,000.
  - Some employee earns over $65,000.
  - Not every female employee earns more than $10,000.

<table>
<thead>
<tr>
<th>Employee</th>
<th>Gender</th>
<th>Salary</th>
<th>Supervisor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al</td>
<td>male</td>
<td>$60,000</td>
<td>-</td>
</tr>
<tr>
<td>Betty</td>
<td>female</td>
<td>$500</td>
<td>Doug</td>
</tr>
<tr>
<td>Carlos</td>
<td>male</td>
<td>$40,000</td>
<td>Ellen</td>
</tr>
<tr>
<td>Doug</td>
<td>male</td>
<td>$30,000</td>
<td>Ellen</td>
</tr>
<tr>
<td>Ellen</td>
<td>female</td>
<td>$50,000</td>
<td>Al</td>
</tr>
<tr>
<td>Flo</td>
<td>female</td>
<td>$20,000</td>
<td>Ellen</td>
</tr>
</tbody>
</table>

Evaluating Existentially Quantified Statements

- To **prove**, give at least **one example** that satisfies the quantification.
- To **disprove**, verify that **every element** of the domain is a counter-example that does not satisfy the quantification.
## Evaluating Quantifiers - Summary

<table>
<thead>
<tr>
<th></th>
<th>Universal</th>
<th>Existential</th>
</tr>
</thead>
<tbody>
<tr>
<td>Verify (prove)</td>
<td>All elements</td>
<td>one example</td>
</tr>
<tr>
<td>Falsify (disprove)</td>
<td>one counter-example</td>
<td>all counter-examples</td>
</tr>
</tbody>
</table>
Chapter 2

Logical Notation

Visualization with Venn Diagram
Visualizing Relationships between Sets

**Venn Diagram**

- The **rectangle** represents the **domain**.
- Each **circle** represents a **set** in the domain.
- **O** in a part of a set means that this part must be **occupied**, i.e., there must be some element in there.
- **X** in a part of a set means that this part must be **empty**, i.e., contains no element.

\[ P \cap Q \neq \emptyset \]
Exercise: Visualizing Relationships between Sets

- Use Venn Diagram to visualize $P \cap Q = \emptyset$. 

![Venn Diagram](image)
Exercise: Visualizing Relationships between Sets

- Use Venn Diagram to visualize $P \subseteq Q$. 

![Venn Diagram](image-url)
Exercise: Visualizing Relationships between Sets

- Use Venn Diagram to visualize the region which represents $P \cup Q \cup R$
Exercise: Visualizing Relationships between Sets

- Use Venn Diagram to visualize \( P \cap Q \cap R = \emptyset \).
Chapter 2

Logical Notation

Logical Sentences and Statements
What is the difference between following sentences?

- The employee makes less than $55,000.
- Betty makes less than $55,000.
- Every employee make less than $55,000.

**Open Sentences vs. Statements**

- **Open Sentences** include *unspecified (unquantified) objects*, and therefore cannot be evaluated.
- All objects in a **closed sentence** (aka **statement**) are either *specified or quantified*, and therefore a statement can be evaluated to True or False.
Exercise: Is it a statement?

- $L(x)$. No
- $\forall x \in E, L(x)$. Yes
- $\forall x \in E, S(x, y)$. No
- Someone took my pen. Yes
- The pen is red. No
- Roses are red. Yes

Transforming open sentences to statements

- **Specifying** the values of unspecified objects:
  
  $L(x) \rightarrow L(\text{Carlos})$

- **Quantifying** over unspecified objects:
  
  $L(x) \rightarrow \forall x, L(x)$
  
  $L(x) \rightarrow \exists x, L(x)$
Chapter 2

Logical Notation

Negation, Conjunction, Disjunction
Predicates

An $n$-ary predicate $L(x_1, ..., x_n)$ is a **boolean function** returning **True** or **False** such that

$$L(x_1, ..., x_n) = \text{True} \text{ if } \langle x_1, ..., x_n \rangle \text{ satisfy the property that is denoted by } L$$

$$L(x_1, ..., x_n) = \text{False} \text{ if } \langle x_1, ..., x_n \rangle \text{ do not satisfy the property that is denoted by } L.$$ 

Example

$$M = \{ Al, Carlos, Doug \}.$$ 

- $M(Al) = \text{True}, M(Carlos) = \text{True}, M(Doug) = \text{True}.$
- $M(Betty) = \text{False}, M(Ellen) = \text{False}, M(Flo) = \text{False}.$
Review: Sets and Predicates

Predicates

An \( n \)-ary predicate \( L(x_1, \ldots, x_n) \) is a \textbf{boolean function} returning \textbf{True} or \textbf{False} such that

\[
L(x_1, \ldots, x_n) = \text{True} \quad \text{if } \quad \langle x_1, \ldots, x_n \rangle \text{ satisfy the property that is denoted by } L
\]

\[
L(x_1, \ldots, x_n) = \text{False} \quad \text{if } \quad \langle x_1, \ldots, x_n \rangle \text{ do not satisfy the property that is denoted by } L.
\]

Important Notes about Predicates

- \( L(x) \) is \textbf{not a set}! In a logical statement, you cannot treat a symbol both as a set and a predicate symbol
  - \textbf{Incorrect} use of notation: \( \forall x, y \in E, x \in L, L(y) \).
  - \textbf{Correct} version: \( \forall x, y \in E, x \in L, y \in L \) or \( \forall x, y \in E, L(x) \land L(y) \).

- Don’t apply \textbf{set operations over predicates}!
  \( P(x) \cap Q(y) \) makes no sense (why?)

- Don’t \textbf{nest} predicates!
  \( P(Q(x)) \) makes no sense (why?)
Negation

Negation Symbol

For the sake of brevity we will write:

- $P(x_1, ..., x_n)$ when $P(x_1, ..., x_n) = \text{True}$
- $\neg P(x_1, ..., x_n)$ when $P(x_1, ..., x_n) = \text{False}$

- “$\neg$“ is called the negation symbol.
- $\neg P(x_1, ..., x_n)$ is the negation of predicate $P(x_1, ..., x_n)$.

Example #1

$F(x)$: $x$ feels good.

Translate the following logical sentence to English

- $\neg F(Betty)$: Betty does not feel good.
- Can we translate $\neg F(Betty)$ to: Betty feels bad?
  - Only if we are given an explicit assumption, or we can formally prove that all elements in the domain either feel good or bad.
Negation

**Negation Symbol**

For the sake of brevity we will write:
- $P(x_1, \ldots, x_n)$ when $P(x_1, \ldots, x_n) = \text{True}$
- $\neg P(x_1, \ldots, x_n)$ when $P(x_1, \ldots, x_n) = \text{False}$

- "$\neg$" is called the **negation symbol**.
- $\neg P(x_1, \ldots, x_n)$ is the negation of predicate $P(x_1, \ldots, x_n)$.

**Example #2**

$M(x)$: $x$ is male.

Translate the following logical sentence to English:
- $\neg M(Betty)$: Betty is **not male**.
- Can we translate $\neg M(Betty)$ to: Betty **is female**?
  - Only if we are given an explicit **assumption**, or we can **formally prove** that all elements in the domain are **either male or female**.
Negation

Negation Symbol

For the sake of brevity we will write:

\( P(x_1, \ldots, x_n) \) when \( P(x_1, \ldots, x_n) = \text{True} \)

\( \neg P(x_1, \ldots, x_n) \) when \( P(x_1, \ldots, x_n) = \text{False} \)

- "\( \neg \)" is called the negation symbol.
- \( \neg P(x_1, \ldots, x_n) \) is the negation of predicate \( P(x_1, \ldots, x_n) \).

Example #3

\( L(x) \): \( x \) earns less than \$55,000.

Translate the following logical sentence to English

- \( L(x) \): \( x \) earns less than \$55,000.
  - \( \neg L(Al) \): \( Al \) does not earn less than \$55,000.
- Can we translate \( \neg L(Al) \) to: \( Al \) earns more than or equal to \$55,000?
  - Yes, because we have the following mathematical fact about numbers:
    For two numbers \( n \) and \( m \), either \( n = m \) or \( n < m \) or \( n > m \).
Conjunction (Logical AND)

**Conjunctive Sentences**

- A **conjunction** is a sentence that joins two other sentences and claims that both of the original sentences are true.
  - Al makes more than $25,000 and less than $75,000.
- **Conjunct Symbol**: ∧
- **Conjunction in logical notation**: $P \land Q$, where $P$ and $Q$ are logical sentences.

$L(x)$: $x$ earns less than $75,000.$
$K(x)$: $x$ earns more than $25,000.$

- Al makes more than $25,000$ and less than $75,000.$
  - $K(Al) \land L(AL)$.
- All employees make more than $25,000$ and less than $75,000.$
  - $\forall x \in E, K(x) \land L(x)$. 

Evaluating Conjunctions

$P \land Q$ is **True** if $P$ is **True** and $Q$ is **True**.

$P \land Q$ is **False** if $P$ is **False** or $Q$ is **False**.

**Evaluating Conjunctions**

- To *prove*, verify that **both** $P$ and $Q$ are **True**.
- To *disprove*, show that **at least one** of $P$ and $Q$ is **False**.
Disjunction (Logical OR)

Disjunctive Sentences

- A **disjunction** is a sentence that joins two other sentences and claims that **at least one** of the original sentences are true.
  - The employee is female **or** makes less than $75,000.

- **Disjunct Symbol**: $\lor$
- **Disjunction in logical notation**: $P \lor Q$, where $P$ and $Q$ are logical sentences.

$L(x)$: $x$ earns less than $75,000$.
$F(x)$: $x$ is female.

- The employee is female or makes less than $75,000$.
  $x \in E, F(x) \lor L(x)$.
- All employees are female or make less than $75,000$.
  $\forall x \in E, F(x) \lor L(x)$.
Evaluating Disjunctions

\[ P \lor Q \text{ is True if } P \text{ is True or } Q \text{ is True.} \]
\[ P \lor Q \text{ is False if } P \text{ is False and } Q \text{ is False.} \]

Evaluating Disjunctions

- To prove, verify that at least one of \( P \) and \( Q \) is True.
- To disprove, show that both \( P \) and \( Q \) are False.
You should be able to understand the following jokes:

Three logicians walk into a bar.
The bartender asks: Do all of you want a drink?
The first logician says: I don’t know.
The second logician says: I don’t know.
The third logician says: Yes!

A logician’s wife is having a baby.
The doctor immediately hands the newborn to the dad.
His wife asks impatiently: So, is it a boy or a girl?
The logician replies: Yes.
(Well, it seems that the logician has made an assumption, right?)