CSC165 Mathematical Expression and Reasoning for Computer Science

Chapter 4: Algorithm Analysis and Asymptotic Notation

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March 4, 2015
Announcements

- Thursday tutorial session will be cancelled for this week.

  Due to large request for office hours, updated TA office hours information for this Thursday is listed below:
  - 12-2 BA3201
  - 4-5:30 BA3201

- Today’s office hour 3-5 will be as usual at BA4261.
Asymptotic notation

- $O$
- $\Omega$
- $\Theta$
Big-O Notation

Here is a precise definition of “The set of functions that are eventually no more than \( f \), to within a constant factor”:

Definition: For any function \( f : \mathbb{N} \to \mathbb{R}_{\geq 0} \) \((i.e., \text{any function mapping naturals to nonnegative reals}), \text{let}

\[
O(f) = \{ g : \mathbb{N} \to \mathbb{R}_{\geq 0} \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq cf(n) \}.
\]

\( g \in O(f) \) means that “\( g \) grows no faster than \( f \)”.
Equivalently, “\( f \) is an upper bound for \( g \)”.

\( \mathbb{R}^+ \): the set of positive real numbers
Definition: For any function $f : \mathbb{N} \rightarrow \mathbb{R}^+ \geq 0$, let

$$\Omega(f) = \{ g : \mathbb{N} \rightarrow \mathbb{R}^+ \geq 0 \mid \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \geq cf(n) \}.$$

“$g \in \Omega(f)$” expresses the concept that “$g$ grows at least as fast as $f$”; $f$ is a lower bound on $g$. 
**Definition:** For any function $f : \mathbb{N} \to \mathbb{R}^\geq 0$, let

$$\Theta(f) = \{ g : \mathbb{N} \to \mathbb{R}^\geq 0 \mid \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n) \}.$$

“$g \in \Theta(f)$” expresses the concept that “$g$ grows at the same rate as $f$”.

$f$ is a tight bound for $g$, or $f$ is both an upper bound and a lower bound on $g$. 
Prove that $2n^3 - 5n^4 + 7n^6$ is in $O(n^2 - 4n^5 + 6n^8)$

We begin with ...

Let $c' = \_\_\_$. Then $c' \in \mathbb{R}^+$. Let $B' = \_\_\_$. Then $B' \in \mathbb{N}$.

Assume $n \in \mathbb{N}$ and $n \geq B'$. 
# arbitrary natural number and antecedent

Then $2n^3 - 5n^4 + 7n^6 \leq \ldots \leq c'(n^2 - 4n^5 + 6n^8)$.

Then $\forall n \in \mathbb{N}, n \geq B' \Rightarrow 2n^3 - 5n^4 + 7n^6 \leq c'(n^2 - 4n^5 + 6n^8)$. 
# introduce $\Rightarrow$
and $\forall$

Hence, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 2n^3 - 5n^4 + 7n^6 \leq c(n^2 - 4n^5 + 6n^8)$. 
# introduce $\exists$
Another $O$ Proof

Prove that $2n^3 - 5n^4 + 7n^6 \in O(n^2 - 4n^5 + 6n^8)$

To fill in the . . .

we try to form a chain of inequalities, working from both ends, simplifying the expressions:

$$2n^3 - 5n^4 + 7n^6 \leq 2n^3 + 7n^6 \quad \text{(drop } -5n^4)$$

$$\leq 2n^6 + 7n^6 \quad \text{(increase } n^3 \text{ to } n^6)$$

$$= 9n^6 \leq 9n^8 \quad \text{(simpler to compare)}$$

$$= 2(9/2)n^8 \quad \text{(choose } c' = 9/2)$$

$$= 2cn^8$$

$$= c'(-4n^8 + 6n^8) \quad \text{(bottom up: decrease } -4n^5 \text{ to } -4n^8)$$

$$\leq c'(-4n^5 + 6n^8) \quad \text{(bottom up: drop } n^2)$$

$$\leq c'(n^2 - 4n^5 + 6n^8)$$

We never needed to restrict $n$ for $n \in \mathbb{N} \ (n \geq 0)$, so we can fill in $c' = 9/2, B' = 0$, and complete the proof.
Here are some general results that we now have the tools to prove.

- $3n^2 + 2n \in O(n^2)$.
- $n^3 \notin O(3n^2)$.
- $2^n \notin O(n^2)$.
- $n^2 + n \in \Omega(15n^2 + 3)$. 
Intuitively, big-Oh notation expresses something about how two functions compare as \( n \) tends toward infinity. But we know of another mathematical notion that captures a similar (though not identical) idea: the concept of *limit*.

**Definition** Let \( f \) be a function defined on some open interval that contains the number \( a \), except possibly at \( a \) itself. Then we say that the limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \), and we write

\[
\lim_{x \to a} f(x) = L
\]

if for every number \( \varepsilon > 0 \) there is a number \( \delta > 0 \) such that

\[
0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon
\]

precisely, recall the following definition, for all \( L \in \mathbb{R}^\geq 0 \):

\[
\lim_{n \to \infty} \frac{f(n)}{g(n)} = L \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow L - \varepsilon < \frac{f(n)}{g(n)} < L + \varepsilon
\]
Calculus: *Limit-1*

Note that [1] can be rewritten as follows:

\[
5 - \varepsilon < f(x) < 5 + \varepsilon
\]

if \(3 - \delta < x < 3 + \delta\) \((x \neq 3)\), then \(5 - \varepsilon < f(x) < 5 + \varepsilon\).

and this is illustrated in Figure 1. By taking the values of \(x \neq 3\) to lie in the interval \((3 - \delta, 3 + \delta)\) we can make the values of \(f(x)\) lie in the interval \((5 - \varepsilon, 5 + \varepsilon)\).

Using [1] as a model, we give a precise definition of a limit.

\[2\quad \text{Definition}\] Let \(f\) be a function defined on some open interval that does not contain a number \(a\), except possibly at \(a\) itself. Then we say that the limit of \(f(x)\) as \(x\) approaches \(a\) is \(L\), and we write

\[
\lim_{x \to a} f(x) = L
\]
Infinite Limits

Infinite limits can also be defined in a precise way. The following is a precise version of Definition 4 in Section 2.2.

6 Definition Let $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that for every positive number $M$ there is a positive number $\delta$ such that

if $0 < |x - a| < \delta$ then $f(x) > M$

precisely, recall the following definition:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \iff \forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \frac{f(n)}{g(n)} > \varepsilon$$
Calculus: *Limit-2*

![Graph showing a function with horizontal line at y = M, and points a - δ and a + δ on the x-axis.](image)
Prove: \( f(n) \in \mathcal{O}(g(n)) \)

Suppose that \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = L \). Intuitively, this tells us that \( \frac{f(n)}{g(n)} \approx L \), for \( n \) “large enough.”

In that case, \( f(n) \approx Lg(n) \) for \( n \) large enough, so we should be able to prove that \( f \in \mathcal{O}(g) \):

Assume \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = L \).

Then \( \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow L - 1.1 < \frac{f(n)}{g(n)} < L + 1.1 \).  

**Definition of limit for \( \varepsilon = 1.1 \)**

Then \( \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \leq (L + 1)g(n) < (L + 1.1)g(n) \).

Then \( f \in \mathcal{O}(g) \).  

**Definition of \( \mathcal{O} \), with \( B = n_0 \) and \( c = L + 1 \)**

Hence, \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = L \Rightarrow f \in \mathcal{O}(g) \).
Prove: \( g(n) \notin O(f(n)) \)

Recall that \( g(n) = 2^n \) and \( f(n) = n \). We rely on the fact that \( \lim_{n \to \infty} \frac{2^n}{n} = \infty \).

Assume \( c \in \mathbb{R}^+ \), assume \( B \in \mathbb{N} \). \# arbitrary values

Then \( \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \frac{2^n}{n} > c \). \# definition of \( \lim_{n \to \infty} \frac{2^n}{n} = \infty \) with \( \varepsilon = c \)

Let \( n_0 \) be such that \( \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \frac{2^n}{n} > c \), and \( n' = \max(B, n_0) \).

Then \( n' \in \mathbb{N} \).

Then \( n' \geq B \). \# by definition of \( \max \)

Then \( 2^{n'} > cn' \) because \( \frac{2^{n'}}{n'} > c \). \# by the first line above, since \( n' \geq n_0 \)

Then \( n' \geq B \land g(n') > cf(n') \). \# introduce \( \land \)

Then \( \exists n \in \mathbb{N}, n \geq B \land g(n) > cf(n) \). \# introduce \( \exists \)

Then \( \forall c \in \mathbb{R}, \forall B \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq B \land g(n) > cf(n) \). \# introduce \( \forall \)

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\(^1\) Applying l'Hôpital's Rule, \( \lim_{n \to \infty} \frac{2^n}{n} = \lim_{n \to \infty} \frac{\ln(2) \cdot 2^n}{1} = \infty \).
Here are some general results that we now have the tools to prove.

- $3n^2 + 2n \in O(n^2)$.
- $n^3 \notin O(3n^2)$.
- $2^n \notin O(n^2)$.
- $n^2 + n \in \Omega(15n^2 + 3)$. 
Here are some general results that we now have the tools to prove.

- $3n^2 + 2n \in \mathcal{O}(n^2)$.
- $n^3 \notin \mathcal{O}(3n^2)$.
- $2^n \notin \mathcal{O}(n^2)$.
- $n^2 + n \in \Omega(15n^2 + 3)$.

- $7n \in \mathcal{O}(n^2)$; $7n \notin \Omega(n^2)$.
- $7n^2 \in \mathcal{O}(n^2)$; $7n^2 \in \Omega(n^2)$; $7n^2 \in \Theta(n^2)$.
- $7n^3 \notin \mathcal{O}(n^2)$; $7n^3 \in \Omega(n^2)$. 
Some Theorems

Here are some general results that we now have the tools to prove.

- $(f \in O(g) \land g \in O(h)) \Rightarrow f \in O(h)$.
  Intuition: If $f$ grows no faster than $g$, and $g$ grows no faster than $h$, then $f$ must grow no faster than $h$. 
Here are some general results that we now have the tools to prove.

- \((f \in O(g) \land g \in O(h)) \Rightarrow f \in O(h)\).
  Intuition: If \(f\) grows no faster than \(g\), and \(g\) grows no faster than \(h\), then \(f\) must grow no faster than \(h\).

- \(g \in \Omega(f) \iff f \in O(g)\).
  Intuition: if \(f\) grows no faster than \(g\), then \(g\) grows no slower than \(f\).
Some Theorems

Here are some general results that we now have the tools to prove.

- \((f \in O(g) \land g \in O(h)) \implies f \in O(h)\).
  Intuition: If \(f\) grows no faster than \(g\), and \(g\) grows no faster than \(h\), then \(f\) must grow no faster than \(h\).

- \(g \in \Omega(f) \iff f \in O(g)\).
  Intuition: if \(f\) grows no faster than \(g\), then \(g\) grows no slower than \(f\).

- \(g \in \Theta(f) \iff g \in O(f) \land g \in \Omega(f)\).
  Intuition: \(g\) grows at the same rate as \(f\). \(f\) is both an upper bound and a lower bound on \(g\).
**Theorem 1**

For any functions $f, g, h : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, we have $(f \in O(g) \land g \in O(h)) \Rightarrow f \in O(h)$.

**Proof:**

Assume $f \in O(g) \land g \in O(h)$.

So $f \in O(g)$.

So $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n > B \Rightarrow f(n) \leq cg(n)$. \footnote{by def'n of $f \in O(g)$}

Let $c_g \in \mathbb{R}^+, B_g \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_g \Rightarrow f(n) \leq c_g g(n)$.

So $g \in O(h)$.

So $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \leq ch(n)$. \footnote{by def'n of $g \in O(h)$}

Let $c_h \in \mathbb{R}^+, B_h \in \mathbb{N}$ be such that $\forall n \in \mathbb{N}, n \geq B_h \Rightarrow g(n) \leq c_h h(n)$.

Let $c' = c_g c_h$. Let $B' = \max(B_g, B_h)$.

Then, $c' \in \mathbb{R}^+$ (because $c_g, c_h \in \mathbb{R}^+$) and $B' \in \mathbb{N}$ (because $B_g, B_h \in \mathbb{N}$).

Assume $n \in \mathbb{N}$ and $n \geq B'$.

Then $n \geq B_h$ (by definition of $\max$), so $g(n) \leq c_h h(n)$.

Then $n \geq B_g$ (by definition of $\max$), so $f(n) \leq c_g g(n) \leq c_g c_h h(n)$.

So $f(n) \leq c'h(n)$.

Hence, $\forall n \in \mathbb{N}, n \geq B' \Rightarrow f(n) \leq c'h(n)$.

Therefore, $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq ch(n)$.

So $f \in O(g)$, by definition.

So $(f \in O(g) \land g \in O(h)) \Rightarrow f \in O(h)$. 

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Theorem 2

For any functions $f, g : \mathbb{N} \to \mathbb{R}^\geq 0$, we have $g \in \Omega(f) \iff f \in \mathcal{O}(g)$.

**Proof:**

\[
g \in \Omega(f) \iff \exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow g(n) \geq cf(n) \quad \text{(by definition)}
\]

\[
\iff \exists c' \in \mathbb{R}^+, \exists B' \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B' \Rightarrow f(n) \leq c'g(n)
\]

(letting $c' = 1/c$ and $B' = B$)

\[
\iff f \in \mathcal{O}(g) \quad \text{(by definition)}
\]
Theorem 3

For any functions \( f, g : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} \), we have \( g \in \Theta(f) \iff g \in \mathcal{O}(f) \land g \in \Omega(f) \).

**Proof:**

\( g \in \Theta(f) \)
\( \iff \) (by definition)
\( \exists c_1 \in \mathbb{R}^+, \exists c_2 \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n). \)
\( \iff \) (combined inequality, and \( B = \max(B_1, B_2) \))
\( \left( \exists c_1 \in \mathbb{R}^+, \exists B_1 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B_1 \Rightarrow g(n) \geq c_1 f(n) \right) \land \)
\( \left( \exists c_2 \in \mathbb{R}^+, \exists B_2 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B_2 \Rightarrow g(n) \leq c_2 f(n) \right) \)
\( \iff \) (by definition)
\( g \in \Omega(f) \land g \in \mathcal{O}(f) \)
Corollary: For any functions $f, g : \mathbb{N} \to \mathbb{R}^{>0}$, we have $g \in \Theta(f) \iff f \in \Theta(g)$.

Proof:

\[
g \in \Theta(f) \iff g \in \mathcal{O}(f) \land g \in \Omega(f) \quad \text{(by 3)}
\]
\[
\iff g \in \mathcal{O}(f) \land f \in \mathcal{O}(g) \quad \text{(by 2)}
\]
\[
\iff f \in \mathcal{O}(g) \land g \in \mathcal{O}(f) \quad \text{(by commutativity of } \land)\]
\[
\iff f \in \mathcal{O}(g) \land f \in \Omega(g) \quad \text{(by 2)}
\]
\[
\iff f \in \Theta(g) \quad \text{(by 3)}
\]