CHAPTER 4

ALGORITHM ANALYSIS AND ASYMPTOTIC NOTATION

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def LS(A, x):
    """Return index i, x == A[i].
    Otherwise, return -1 """
    1. i = 0
    2. while i < len(A):
    3.     if A[i] == x:
    4.         return i
    5.     i = i + 1
    6. return -1

What is the runtime of LS(A, x)?
if the first index where x is found is $k$
i.e., $A[k] == x$

$t_{LS}(A, x) = 1 + 3(k+1)$
$= 3k + 4$

$t_{LS}([2, 4, 6, 8], 6) = 10$
Today’s Outline

- Formal definition of $O$, $\Omega$, $\Theta$
FORMAL DEFINITIONS OF $O, \Omega, \Theta$
Recap $O(n^2)$

Set of functions that **grow no faster** than $n^2$
- count the number of steps
- constant factors don’t matter
- only highest-order term matter

The following functions are in $O(n^2)$

\[
n^2 \quad 2n^2 + 3n \quad \frac{n^2}{165} + 1130n + 3.14159
\]
Formal definition of $O(n^2)$

A function $f(n)$ is in $O(n^2)$ iff

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{ such that } \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2$$

**Beyond breakpoint $B$, $f(n)$ is upper-bounded by $cn^2$, where $c$ is some wisely chosen constant multiplier.**
A chicken grows slower than a turkey in the sense that, after a certain breakpoint, a chicken will always be smaller than a turkey.
Formal Definition $O(n^2)$

A function $f(n)$ is in $O(n^2)$ iff

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{ such that } \forall n \in \mathbb{N}, n \geq B \Rightarrow f(n) \leq cn^2$$

**Simple example: prove** $700n^2 \in O(n^2)$

Pick $c = 711$, or any real number $\geq 700$

Pick $B = 0$, or any natural number $\geq 0$

then $\forall n \in \mathbb{N}, n \geq 0 \Rightarrow 700n^2 \leq 711n^2$

then $\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq B \Rightarrow 700n^2 \leq cn^2$

then $700n^2 \in O(n^2)$
Formal Definition $\Omega(n^2)$

A function $f(n)$ is in $O(n^2)$ if and only if

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{ such that } \forall n \in \mathbb{N}, \ n \geq B \Rightarrow f(n) \leq cn^2$$

A function $f(n)$ is in $\Omega(n^2)$ if and only if

$$\exists c \in \mathbb{R}^+, \exists B \in \mathbb{N}, \text{ such that } \forall n \in \mathbb{N}, \ n \geq B \Rightarrow f(n) \geq cn^2$$

$O(n^2)$: set of functions that grow no faster than $n^2$

$\Omega(n^2)$: set of functions that grow no slower than $n^2$

$\Theta(n^2)$: set of functions that are in both $O(n^2)$ and $\Omega(n^2)$ (functions growing as fast as $n^2$)
Growth rate ranking of some common functions

\[ f(n) = n^n \]
\[ f(n) = 2^n \]
\[ f(n) = n^3 \]
\[ f(n) = n^2 \]
\[ f(n) = n \log n \]
\[ f(n) = \sqrt{n} \]
\[ f(n) = \log n \]
\[ f(n) = 1 \]
Examples

\[ 7n \in \mathcal{O}(n^2) \quad 7n \notin \Omega(n^2) \]

\[ 7n^3 \notin \mathcal{O}(n^2) \quad 7n^3 \in \Omega(n^2) \]

\[ 7n^2 \in \mathcal{O}(n^2) \quad 7n^2 \in \Omega(n^2) \]

\[ 7n^2 \in \Theta(n^2) \]
Next Week

• Worst-case analysis of two algorithms