CSC 236
St George Campus

University of Toronto

Assignment #4
Due: November 18, 2004, by 6:00 PM
(in the CSC 236 drop box in BA 2220)

Question 1. (10 marks) Consider the first-order language of arithmetic described in Section 6.2 of the notes. Let $\mathcal{N}$ and $\mathcal{Z}$ be structures for this language, with domains $\mathbb{N}$ and $\mathbb{Z}$, respectively, and the standard meaning for the predicate symbols. More formally:

$$
\begin{array}{ll}
S^{\mathcal{N}} = \{(a,b,c) \in \mathbb{N}^3 : a+b=c\} & S^{\mathcal{Z}} = \{(a,b,c) \in \mathbb{Z}^3 : a+b=c\} \\
P^{\mathcal{N}} = \{(a,b,c) \in \mathbb{N}^3 : a \cdot b=c\} & P^{\mathcal{Z}} = \{(a,b,c) \in \mathbb{Z}^3 : a \cdot b=c\} \\
L^{\mathcal{N}} = \{(a,b) \in \mathbb{N}^2 : a<b\} & L^{\mathcal{Z}} = \{(a,b) \in \mathbb{Z}^2 : a<b\} \\
\approx^{\mathcal{N}} = \{(a,b) \in \mathbb{N}^2 : a=b\} & \approx^{\mathcal{Z}} = \{(a,b) \in \mathbb{Z}^2 : a=b\} \\
\boldsymbol{0}^{\mathcal{N}} = 0 & \boldsymbol{0}^{\mathcal{Z}} = 0 \\
\boldsymbol{1}^{\mathcal{N}} = 1 & \boldsymbol{1}^{\mathcal{Z}} = 1
\end{array}
$$

For each of the sentences below, state whether it is true or false in each of $\mathcal{N}$ and $\mathcal{Z}$. Justify your answer by translating the formula into a statement (in precise English) about numbers, and then explain why that statement is true or false for natural numbers and for integers.

(a) $\forall x\, \exists y\, \exists z\, \big(S(y,z,x) \wedge \neg{\approx}(y,z)\big)$

(b) $\exists x\, \forall y\, \big(\neg{\approx}(y,x) \to L(y,x)\big)$

(c) $\forall x\, \forall y\, \Big(\big(L(x,0) \wedge P(x,x,y)\big) \to L(0,y)\Big)$

(d) $\forall x\, \forall y\, \forall z\, \forall u\, \forall v\, \Big(\big(L(x,y) \wedge \neg{\approx}(z,\boldsymbol{0}) \wedge P(x,z,u) \wedge P(y,z,v)\big) \to L(u,v)\Big)$

(e) $\exists x\, \forall y\, \big(L(y,x) \to \exists x\, S(x,\boldsymbol{1},x)\big)$

Question 2. (10 marks) For each of the following claims, state whether it is true or false, and justify your answer. In the formulas involved in these claims, A and B are unary predicate symbols in the first-order language; and E and F are arbitrary first-order formulas.

a. $\neg\forall x\, \big(A(x) \to B(x)\big)$ is logically equivalent to $\exists x\, \big(A(x) \wedge \neg B(x)\big)$.

b. $\exists x\, A(x) \to \exists x\, B(x)$ is logically equivalent to $\exists x\, \big(A(x) \to B(x)\big)$.

c. $\exists x\, A(x) \vee \exists x\, \neg A(x)$ is logically equivalent to $\forall x\, \big(A(x) \vee \neg A(x)\big)$.

d. $\forall x\, A(x) \vee \forall x\, \neg A(x)$ is logically equivalent to $\forall x\, \big(A(x) \vee \neg A(x)\big)$.

e. If x does not appear free in F then $\forall x\, E \oplus F$ is logically equivalent to $\forall x\, (E \oplus F)$.

continued

Question 3. (10 marks)

a. (5 marks) Abraham Lincoln is credited with the famous statement: "You can fool all of the people some of the time, and you can fool some of the people all of the time, but you can't fool all of the people all of the time."

Consider a first-order language with unary predicate symbols P and T, and binary predicate symbol F with the following intended interpretations: $P(x)$ means "x is a person", $T(x)$ means "x is a time"; and $F(x, y)$ means "you can fool x at y". Translate each of the statements below into a formula in this language:

(i) You can fool all of the people some of the time.

(ii) You can fool some of the people all of the time.

(iii) You can't fool all of the people all of the time.

Statements (i) and (ii) are ambiguous. For each of them, provide formulas for both meanings and indicate which meaning you believe Lincoln intended.

Note: The domain of interpretation for this problem contains "objects" of two sorts: people and times. The predicates P and T are used to indicate the sort of a variable. For example, $P(x)$ indicates that variable x refers to a person.

b. (5 marks) There is a jazz tune with the colourful title, "everybody loves my baby but my baby don't love nobody but me" or, in somewhat more "standard" English:

Everybody loves my baby but my baby loves nobody but me. $(*)$

(i) Translate $(*)$ into a formula of the first-order language that contains two constant symbols: $\boldsymbol{i}$ (intended to stand for the person to whom the words "my" and "me" refer in $(*)$), and $\boldsymbol{b}$ (intended to stand for the person referred to as "my baby" in $(*)$); and two binary predicate symbols: $L(x, y)$ (intended to stand for the predicate "x loves y"), and $\approx(x, y)$ (which stands for equality — i.e., "x is the same as y").

(ii) Give a careful argument in English proving that $(*)$ logically implies that my baby is me — i.e., that $\boldsymbol{i}$ and $\boldsymbol{b}$ are the same!

Question 4. (10 marks) Using only the logical equivalences listed in Chapters 5 and 6 of the notes, transform the following first-order formula into an equivalent Prenex Normal Form (PNF) formula. Demonstrate the steps through which you obtain your final formula.

a. $\forall x\, R(x) \to \big(\exists x\, P(x) \lor \exists x\, Q(x)\big)$

b. $\neg\forall x\, S(x, y) \leftrightarrow \exists y\, T(x, y, z)$

continued

Question 5. (10 marks) Consider a relational database that describes an aspect of the activities in a university. Specifically, the database schema consists of the following relations:

- $\mathit{Student}(s, n, a)$ — a tuple (s, n, a) belongs to this relation if the student whose SIN is s has name n and address a. We assume that the SIN uniquely identifies a student; however, there could be different students with the same name (or address).
- $\mathit{Course}(c, t)$ — a tuple (c, t) belongs to this relation if there is a course whose code is c (e.g., CSCB36) and whose title is t (e.g., Introduction to the Theory of Computation). We assume that the course code uniquely identifies a course.
- $\mathit{Took}(s, c, y, m)$ — a tuple (s, c, y, m) belongs to this relation if the student whose SIN is s took course c in year y and received a mark of m. We assume that each course is offered at most once each year.
- $\mathit{Taught}(p, c, y)$ — a tuple (p, c, y) belongs to this relation if the professor whose name is p taught (or is now teaching) course with code c in year y. We assume that each professor is uniquely identified by her/his name.

Write formulas to express the following queries. (Your formulas may use only the predicates listed above and equality.)

a. Find the titles of all courses that Euclid ever took.

b. Find the title of every course that Euclid ever took in which the instructor (the year that Euclid took the course) was Pythagoras.

c. Find the name and address of every classmate of Euclid in a course in which the classmate got an A but Euclid did not.

d. Find the course id and title of every course that has been taught by Pythagoras and only by Pythagoras.

e. Find every year in which Pythagoras taught (the course whose title is) Geometry and no student failed any course taught by Pythagoras that year.