CSC 236 — September 23, 2004
St George Campus — University of Toronto

Assignment #1

Due: October 7, 2004, by 6:00 PM
(in the drop box in BA 2220)

Please re-read the statement about collaboration versus plagiarism on the course information sheet!

Question 1. (5 marks) Use induction to prove that for every integer $n > 0$,
$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \quad \leq \quad 2\sqrt{n} - 1.$

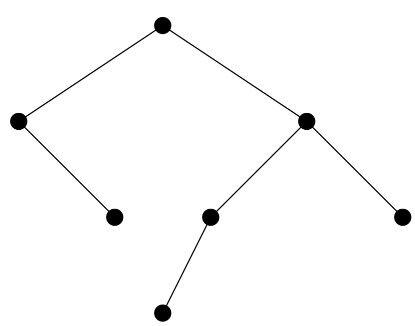
Question 2. (10 marks) Use induction to prove that for every $n \in \mathbb{N}$, $7^n - 2^n$ is a multiple of 5.

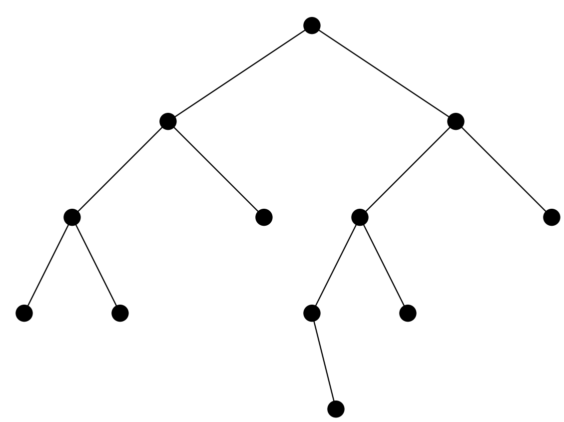
Question 3. (10 marks) The height of a non-empty tree is defined as the maximum number of *edges* on any path that starts at the root of the tree and ends at a leaf. (Thus, the height of a tree with a single node is zero.)

A binary tree is called ***height-balanced*** if every internal (i.e., non-leaf) node u of the tree has the following property:

- If u has only one child, then the subtree rooted at that child has height 0; if u has two children, then the heights of the subtrees rooted at those children differ by at most one.

For example, of the two trees below, the one on the left is height-balanced, while the one on the right is not.





(As discussed in other courses, height-balanced trees play an important role in dynamic data structures that support efficient searching.)

a. (9 marks) Prove that for all $h \in \mathbb{N}$, any height-balanced tree of height h has at least 1.6^h nodes.

b. (1 marks) Is it true that for all $h \in \mathbb{N}$, any binary tree of height h has at least 1.6^h nodes? Prove the validity of your answer.

continued

Question 4. (10 marks) Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined recursively as follows:

$$f(n) = \begin{cases} 10, & \text{if } n = 0 \\ 6, & \text{if } n = 1 \\ 3, & \text{if } n = 2 \\ f(n-3) + n, & \text{if } n > 2 \end{cases}$$

Find the smallest natural number d such that for all integers $n \geq d$, $f(n) \leq n^2/2$. Prove the validity of your answer.

Question 5. (10 marks) Nirvana Universal Trucking (NUT) provides transportation service between $n \geq 1$ cities, using trucks. Each NUT truck is dedicated to transporting goods back and forth between a specific pair of the n cities. Only some pairs of cities have NUT trucks running between them, but it is possible to ship goods from any one of the cities to any other — possibly going through some intermediate cities — using only NUT trucks. Prove that NUT has *at least* $n - 1$ trucks.

Hint: Consider a set of $n \geq 2$ cities interconnected by NUT trucks as described above. If we remove one city and every NUT truck that operates between this city and some other city, the remaining $n - 1$ cities can be grouped into a collection of one or more disjoint sets of cities so that: (a) we can transport goods between cities within the same set using only NUT trucks, but (b) we cannot transport goods between cities in different sets using only NUT trucks (because the only way to do so is via the city that was removed).